

INVERSE SCATTERING FOR THE MAGNETIC SCHRÖDINGER OPERATOR ON SURFACES WITH EUCLIDEAN ENDS

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ABSTRACT. We prove a fixed frequency inverse scattering result for the magnetic Schrödinger operator (or connection Laplacian) on surfaces with Euclidean ends. We show that, under suitable decaying conditions, the scattering matrix for the operator determines both the gauge class of the connection and the zeroth order potential.

1. INTRODUCTION

The purpose of this paper is to show that the scattering matrix $S_{X,V}(\lambda)$ of the magnetic Schrödinger operator $(d + iX)^*(d + iX) + V$ determines V and the gauge class of the 1-form X on Riemann surfaces with Euclidean ends.

When $X = 0$ this was done by in [11] and we refer the reader to the article for all the relevant references and results in this case.

In dimensions $n \geq 3$ the problem of scattering by the magnetic Schrödinger operator was first considered in the simply connected setting by [6] in the smooth case and later by [25] for less regular coefficients. As the setting is Euclidean, determining the gauge of X is equivalent to determining its exterior derivative. Some cohomological aspects of this problem was considered in [2, 3] which described the Aharonov-Bohm effect using inverse scattering. These works still take place in the Euclidean setting and the topology is obtained by removing open balls from \mathbb{R}^n . Observe that when one assumes the coefficients are compactly supported, the inverse scattering problem is equivalent to the Calderón problem on the domain of support and this was done in [21] for $n \geq 3$ and [13, 10] for $n = 2$.

In the present work we focus on the more geometric aspect of the problem where the ambient manifold is a general Riemann surface with Euclidean ends. We prove the following theorem

Theorem 1.1. Let (M_0, g_0) be a non-compact Riemann surface with genus \mathcal{G} and N ends isometric to $\mathbb{R}^2 \setminus \{|z| \leq 1\}$ with metric $|dz|^2$. Let $V_1, V_2 \in C^{1,\beta}(M_0)$ be two potentials with $\beta > 0$ and $X_1, X_2 \in H^{3+\epsilon_0}(M_0)$ for some $\epsilon_0 > 0$ be two 1-forms such that $S_{X_1,V_1}(\lambda) = S_{X_2,V_2}(\lambda)$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Let $d(z, z_0)$ denote the distance between z and a fixed point $z_0 \in M_0$. If $N \geq \max(2\mathcal{G}+1, 2)$, $V_j \in e^{-\gamma d(\cdot, z_0)} L^\infty(M_0)$, and $X_j \in e^{-\gamma d(\cdot, z_0)} H^{3+\epsilon_0}(M_0)$ for all $\gamma > 0$, then there exists a unitary function $\Theta \in 1 + e^{-\gamma d(\cdot, z_0)} W^{1,\infty}(M_0)$

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for all $\gamma > 0$ such that $X_1 - X_2 = d\Theta/\Theta$ and $V_1 = V_2$.

The approach we take in dealing with the non-trivial magnetic term X_j is by viewing $d + iX_j$ as a unitary connection acting on the trivial line bundle over M_0 . Composition with the projection $\pi_{0,1} : T^*M_0 \rightarrow T_{0,1}^*M_0$ yields a Cauchy-Riemann operator $(\bar{\partial} + iA_j) := \pi_{0,1}(d + iX)$ which, by [17], yields a unique complex structure that can be trivialized by choosing a non-vanishing holomorphic section F_{A_j} .

The advantage of such trivializations is that it acts as a bridge between the two Cauchy-Riemann operators $(\bar{\partial} + iA_1)$ and $(\bar{\partial} + iA_2)$. Intuitively, this would effectively reduce the problem to the simpler case where $X_1 = X_2 = 0$ and one can then apply the techniques of [11]. Unfortunately, it is not always true that this conjugation between complex structures preserves the scattering information at infinity. We will see, in fact, that one can judiciously choose the trivializations to preserve this information precisely when the scattering matrices agree. It will become apparent that this is a global result about the trivialization rather than a local one of determining the asymptotic expansions of certain coefficients at infinity. In this sense the "boundary determination" performed here is quite different from those in [39, 16, 15].

This approach to treating Cauchy-Riemann operators was explored in [34], [1], and [10] for studying inverse boundary value problems. The setting of this article, however, is on a non-compact surface and therefore the previously used techniques for Calderón problems do not immediately apply.

The technique used here is the machinery of the b-calculus developed by Melrose in [19]. We will show that the condition $S_{X_1, V_2}(\lambda) = S_{X_2, V_2}(\lambda)$ induces an orthogonality relation between the difference of the trivializations and antiholomorphic 1-forms which belong to certain weighted L^2 spaces. Due to the results of b-calculus we can invoke Fredholm theory to show the existence of a holomorphic function which has the same expansion near infinity as the difference of the trivializations.

In addition to this complication, the presence of a first-order term requires the construction of a different type of CGO and a different integral identity than those used in [11]. Particularly, in order to recover the gauge class of X we will construct a class of CGOs which is compatible with the new boundary integral identity

$$\int_{M_0} \langle (|F_{A_1}|^{-2} - |F_{A_2}|^{-2})\bar{\partial}\tilde{v}, \bar{\partial}\tilde{w} \rangle + \frac{1}{2} \langle (Q_1|F_{A_1}|^2 - Q_2|F_{A_2}|^2)\tilde{v}, \tilde{w} \rangle = 0,$$

relating the size of the two trivializations $|F_{A_j}|$. The modulus of the trivializations turns out to carry all the information we need to recover the gauge class (see Proof of Proposition 8.1).

The organization of this article is as follows. In Section 2 we prove general facts about holomorphic functions and Fredholm properties of the Cauchy-Riemann operators on weighted spaces. Section 3 will be devoted to Carleman estimates on weighted spaces which can produce higher regularity solvability results than those in [11]. In Section 4 we develop the scattering

theory for the magnetic Schrödinger operator and construct the scattering matrix. We will show in Section 5 that the scattering matrix determines the asymptotic behaviour of the trivialization of the Cauchy-Riemann operator. This asymptotic behaviour will be exploited in Section 7 when we derive a new integral identity which is more suitable for recovering the gauge class. In Section 6 we will construct CGOs which we will then use in Sections 8 and 9 to recover the desired information.

2. HOLOMORPHIC MORSE FUNCTIONS ON A SURFACE WITH EUCLIDEAN ENDS

2.1. Riemann surfaces with Euclidean ends. The contents of this section are similar to that in [11]. We include it here only for the convenience of the reader.

Let (M_0, g_0) be a non-compact connected smooth Riemannian surface with N ends E_1, \dots, E_N which are Euclidean, i.e. isometric to $\mathbb{C} \setminus \{|z| \leq 1\}$ with metric $|dz|^2$. By using a complex inversion $z \rightarrow 1/z$, each end is also isometric to a pointed disk

$$E_i \simeq \{|z| \leq 1, z \neq 0\} \text{ with metric } \frac{|dz|^2}{|z|^4}$$

thus conformal to the Euclidean metric on the pointed disk. The surface M_0 can then be compactified by adding the points corresponding to $z = 0$ in each pointed disk corresponding to an end E_i , we obtain a closed Riemann surface M with a natural complex structure induced by that of M_0 , or equivalently a smooth conformal class on M induced by that of M_0 . Another way of thinking is to say that M_0 is the closed Riemann surface M with N points e_1, \dots, e_N removed. The Riemann surface M has holomorphic charts $z_\beta : U_\beta \rightarrow \mathbb{C}$ and we will denote by z_1, \dots, z_N the complex coordinates corresponding to the ends of M_0 , or equivalently to the neighbourhoods of the points e_i . The Hodge star operator \star acts on the cotangent bundle T^*M , its eigenvalues are $\pm i$ and the respective eigenspaces $T_{1,0}^*M := \ker(\star + i\text{Id})$ and $T_{0,1}^*M := \ker(\star - i\text{Id})$ are sub-bundles of the complexified cotangent bundle $\mathbb{C}T^*M$ and the splitting $\mathbb{C}T^*M = T_{1,0}^*M \oplus T_{0,1}^*M$ holds as complex vector spaces. Since \star is conformally invariant on 1-forms on M , the complex structure depends only on the conformal class of g . In holomorphic coordinates $z = x + iy$ in a chart U_β , one has $\star(udx + vdy) = -vdx + udy$ and

$$T_{1,0}^*M|_{U_\beta} \simeq \mathbb{C}dz, \quad T_{0,1}^*M|_{U_\beta} \simeq \mathbb{C}d\bar{z}$$

where $dz = dx + idy$ and $d\bar{z} = dx - idy$. We define the natural projections induced by the splitting of $\mathbb{C}T^*M$

$$\pi_{1,0} : \mathbb{C}T^*M \rightarrow T_{1,0}^*M, \quad \pi_{0,1} : \mathbb{C}T^*M \rightarrow T_{0,1}^*M.$$

The exterior derivative d defines the de Rham complex $0 \rightarrow \Lambda^0 \rightarrow \Lambda^1 \rightarrow \Lambda^2 \rightarrow 0$ where $\Lambda^k := \Lambda^k T^*M$ denotes the real bundle of k -forms on M . Let us denote $\mathbb{C}\Lambda^k$ the complexification of Λ^k , then the ∂ and $\bar{\partial}$ operators can be defined as differential operators $\partial : \mathbb{C}\Lambda^0 \rightarrow T_{1,0}^*M$ and $\bar{\partial} : \mathbb{C}\Lambda^0 \rightarrow T_{0,1}^*M$ by

$$\partial f := \pi_{1,0}df, \quad \bar{\partial} f := \pi_{0,1}df,$$

they satisfy $d = \partial + \bar{\partial}$ and are expressed in holomorphic coordinates by

$$\partial f = \partial_z f dz, \quad \bar{\partial} f = \partial_{\bar{z}} f d\bar{z},$$

with $\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$. Similarly, one can define the ∂ and $\bar{\partial}$ operators from $\mathbb{C}\Lambda^1$ to $\mathbb{C}\Lambda^2$ by setting

$$\partial(\omega_{1,0} + \omega_{0,1}) := d\omega_{0,1}, \quad \bar{\partial}(\omega_{1,0} + \omega_{0,1}) := d\omega_{1,0}$$

if $\omega_{0,1} \in T_{0,1}^*M$ and $\omega_{1,0} \in T_{1,0}^*M$. In coordinates this is simply

$$\partial(udz + vd\bar{z}) = \partial v \wedge d\bar{z}, \quad \bar{\partial}(udz + vd\bar{z}) = \bar{\partial}u \wedge dz.$$

If g is a metric on M whose conformal class induces the complex structure $T_{1,0}^*M$, there is a natural operator, the Laplacian acting on functions and defined by

$$\Delta f := -2i \star \bar{\partial} \partial f = d^* d$$

where d^* is the adjoint of d through the metric g and \star is the Hodge star operator mapping Λ^2 to Λ^0 and induced by g as well.

2.2. Holomorphic functions. We are going to construct Carleman weights given by holomorphic functions on M_0 which grow at most linearly or quadratically in the ends. We will use the Riemann-Roch theorem, following ideas of [9], however, the difference in the present case is that we have very little freedom to construct these holomorphic functions, simply because there is just a finite dimensional space of such functions by Riemann-Roch. For the convenience of the reader, and to fix notations, we recall the usual Riemann-Roch index theorem (see Farkas-Kra [7] for more details). A divisor D on M is an element

$$D = ((p_1, n_1), \dots, (p_k, n_k)) \in (M \times \mathbb{Z})^k, \text{ where } k \in \mathbb{N}$$

which will also be denoted $D = \prod_{i=1}^k p_i^{n_i}$ or $D = \prod_{p \in M} p^{\beta(p)}$ where $\beta(p) = 0$ for all p except $\beta(p_i) = n_i$. The inverse divisor of D is defined to be $D^{-1} := \prod_{p \in M} p^{-\beta(p)}$ and the degree of the divisor D is defined by $\deg(D) := \sum_{i=1}^k n_i = \sum_{p \in M} \beta(p)$. A non-zero meromorphic function on M is said to have divisor D if $(f) := \prod_{p \in M} p^{\text{ord}(p)}$ is equal to D , where $\text{ord}(p)$ denotes the order of p as a pole or zero of f (with positive sign convention for zeros). Notice that in this case we have $\deg(f) = 0$. For divisors $D' = \prod_{p \in M} p^{\beta'(p)}$ and $D = \prod_{p \in M} p^{\beta(p)}$, we say that $D' \geq D$ if $\beta'(p) \geq \beta(p)$ for all $p \in M$. The same exact notions apply for meromorphic 1-forms on M . Then we define for a divisor D

$$r(D) := \dim(\{f \text{ meromorphic function on } M; (f) \geq D\} \cup \{0\}),$$

$$i(D) := \dim(\{u \text{ meromorphic 1 form on } M; (u) \geq D\} \cup \{0\}).$$

The Riemann-Roch theorem states the following identity: for any divisor D on the closed Riemann surface M of genus \mathcal{G} ,

$$r(D^{-1}) = i(D) + \deg(D) - \mathcal{G} + 1. \quad (1)$$

Notice also that for any divisor D with $\deg(D) > 0$, one has $r(D) = 0$ since $\deg(f) = 0$ for all f meromorphic. By [7, Th. p70], let D be a divisor, then

for any non-zero meromorphic 1-form ω on M , one has

$$i(D) = r(D(\omega)^{-1}) \quad (2)$$

which is thus independent of ω . For instance, if $D = 1$, we know that the only holomorphic function on M is 1 and one has $1 = r(1) = r((\omega)^{-1}) - \mathcal{G} + 1$ and thus $r((\omega)^{-1}) = \mathcal{G}$ if ω is a non-zero meromorphic 1 form. Now if $D = (\omega)$, we obtain again from (1)

$$\mathcal{G} = r((\omega)^{-1}) = 2 - \mathcal{G} + \deg((\omega))$$

which gives $\deg((\omega)) = 2(\mathcal{G} - 1)$ for any non-zero meromorphic 1-form ω . In particular, if D is a divisor such that $\deg(D) > 2(\mathcal{G} - 1)$, then we get $\deg(D(\omega)^{-1}) = \deg(D) - 2(\mathcal{G} - 1) > 0$ and thus $i(D) = r(D(\omega)^{-1}) = 0$, which implies by (1)

$$\deg(D) > 2(\mathcal{G} - 1) \implies r(D^{-1}) = \deg(D) - \mathcal{G} + 1 \geq \mathcal{G}. \quad (3)$$

Now we deduce the

Lemma 2.1. Let e_1, \dots, e_N be distinct points on a closed Riemann surface M with genus \mathcal{G} , and let z_0 be another point of $M \setminus \{e_1, \dots, e_N\}$. If $N \geq \max(2\mathcal{G} + 1, 2)$, the following hold true:

- (i) there exists a meromorphic function f on M with at most simple poles, all contained in $\{e_1, \dots, e_N\}$, such that $\partial f(z_0) \neq 0$,
- (ii) there exists a meromorphic function f on M with at most simple poles, all contained in $\{e_1, \dots, e_N\}$, such that z_0 is a zero of order at least 2 of f .
- (iii) there exists a meromorphic function f whose (non-removable) poles are all simple and form precisely the set $\{e_1, \dots, e_N\}$.

Proof. Let first $\mathcal{G} \geq 1$, so that $N \geq 2\mathcal{G} + 1$. By the discussion before the Lemma, we know that there are at least $\mathcal{G} + 2$ linearly independent (over \mathbb{C}) meromorphic functions $f_0, \dots, f_{\mathcal{G}+1}$ on M with at most simple poles, all contained in $\{e_1, \dots, e_{2\mathcal{G}+1}\}$. Without loss of generality, one can set $f_0 = 1$ and by linear combinations we can assume that $f_1(z_0) = \dots = f_{\mathcal{G}+1}(z_0) = 0$. Now for (ii) consider the divisor $D_j = e_1 \dots e_{2\mathcal{G}+1} z_0^{-j}$ for $j = 1, 2$, with degree $\deg(D_j) = 2\mathcal{G} + 1 - j$, then by the Riemann-Roch formula (more precisely (3))

$$r(D_j^{-1}) = \mathcal{G} + 2 - j.$$

Thus, since $r(D_1^{-1}) > r(D_2^{-1}) = \mathcal{G}$ and using the assumption that $\mathcal{G} \geq 1$, we deduce that there is a function in $\text{span}(f_1, \dots, f_{\mathcal{G}+1})$ which has a zero of order 2 at z_0 and a function which has a zero of order exactly 1 at z_0 . To show (iii) observe that if $N \geq 2\mathcal{G} + 1$ then $r((e_1 \dots e_N)^{-1}) = N - \mathcal{G} + 1$. Suppose none of the meromorphic functions with divisor greater than or equal to $(e_1 \dots e_N)^{-1}$ has a pole at e_N then one would have that $r((e_1 \dots e_{N-1})^{-1}) = N - \mathcal{G} + 1$. But $\deg(e_1 \dots e_{N-1}) = N - 1 \geq 2\mathcal{G} - 1$ so $r((e_1 \dots e_{N-1})^{-1}) = N - \mathcal{G}$ by (3). This is a contradiction and therefore every point of $\{e_1, \dots, e_N\}$ is a pole for some meromorphic function with divisor greater than or equal to $(e_1 \dots e_N)^{-1}$. Taking suitable linear combination of these functions yields a meromorphic function with simple poles precisely at the points $\{e_1, \dots, e_N\}$.

The same method clearly works if $\mathcal{G} = 0$ by taking $N \geq 2$. \square

2.3. Morse holomorphic functions with prescribed critical points.

We follow in this section the arguments used in [9] to construct holomorphic functions with non-degenerate critical points (i.e. Morse holomorphic functions) on the surface M_0 with genus \mathcal{G} and N ends, such that these functions have at most linear growth in the ends if $N \geq \max(2\mathcal{G} + 1, 2)$. We let \mathcal{H} be the complex vector space spanned by the meromorphic functions on M with divisors larger or equal to $e_1^{-1} \dots e_N^{-1}$ where $e_1, \dots, e_N \in M$ are points corresponding to the ends of M_0 as explained in the previous section. Note that \mathcal{H} is a complex vector space of complex dimension greater or equal to $N - \mathcal{G} + 1$ for the $e_1^{-1} \dots e_N^{-1}$ divisor. We will also consider the real vector space H spanned by the real parts and imaginary parts of functions in \mathcal{H} , this is a real vector space which admits a Lebesgue measure. We now prove the following

Lemma 2.2. The set of functions $u \in H$ which are not Morse in M_0 has measure 0 in H , in particular its complement is dense in H .

Proof. We use an argument very similar to that used by Uhlenbeck [35]. We start by defining $m : M_0 \times H \rightarrow T^*M_0$ by $(p, u) \mapsto (p, du(p)) \in T_p^*M_0$. This is clearly a smooth map, linear in the second variable, moreover $m_u := m(\cdot, u) = (\cdot, du(\cdot))$ is smooth on M_0 . The map u is a Morse function if and only if m_u is transverse to the zero section, denoted $T_0^*M_0$, of T^*M_0 , i.e. if

$$\text{Image}(D_p m_u) + T_{m_u(p)}(T_0^*M_0) = T_{m_u(p)}(T^*M_0), \quad \forall p \in M_0 \text{ such that } m_u(p) = (p, 0).$$

This is equivalent to the fact that the Hessian of u at critical points is non-degenerate (see for instance Lemma 2.8 of [35]). We recall the following transversality result, the proof of which is contained in [35, Th.2] by replacing Sard-Smale theorem by the usual finite dimensional Sard theorem:

Theorem 2.1. Let $m : X \times H \rightarrow W$ be a C^k map and X, W be smooth manifolds and H a finite dimensional vector space, if $W' \subset W$ is a submanifold such that $k > \max(1, \dim X - \dim W + \dim W')$, then the transversality of the map m to W' implies that the complement of the set $\{u \in H; m_u \text{ is transverse to } W'\}$ in H has Lebesgue measure 0.

We want to apply this result with $X := M_0$, $W := T^*M_0$ and $W' := T_0^*M_0$, and with the map m as defined above. We have thus proved our Lemma if one can show that m is transverse to W' . Let (p, u) such that $m(p, u) = (p, 0) \in W'$. Then identifying $T_{(p,0)}(T^*M_0)$ with $T_p M_0 \oplus T_p^* M_0$, one has

$$Dm_{(p,u)}(z, v) = (z, dv(p) + \text{Hess}_p(u)z)$$

where $\text{Hess}_p(u)$ is the Hessian of u at the point p , viewed as a linear map from $T_p M_0$ to $T_p^* M_0$ (note that this is different from the covariant Hessian defined by the Levi-Civita connection). To prove that m is transverse to W' we need to show that $(z, v) \rightarrow (z, dv(p) + \text{Hess}_p(u)z)$ is onto from $T_p M_0 \oplus H$ to $T_p M_0 \oplus T_p^* M_0$, which is realized if the map $v \rightarrow dv(p)$ from H to $T_p^* M_0$ is onto. But from Lemma 2.1, we know that there exists a meromorphic function f with real part $v = \text{Re}(f) \in H$ such that $v(p) = 0$ and $dv(p) \neq 0$ as an element of $T_p^* M_0$. We can then take $v_1 := v$ and $v_2 := \text{Im}(f)$, which are functions of H such that $dv_1(p)$ and $dv_2(p)$ are linearly independent in

$T_p^*M_0$ by the Cauchy-Riemann equation $\bar{\partial}f = 0$. This shows our claim and ends the proof by using Theorem 2.1. \square

In particular, by the Cauchy-Riemann equation, Lemma 2.2 implies

Corollary 2.1. The subset of functions in \mathcal{H} which are Morse is dense.

This shows, in conjunction with part (iii) of Lemma 2.1 that

Lemma 2.3. Let e_1, \dots, e_N be distinct points on a closed Riemann surface M with genus \mathcal{G} . If $N \geq \max(2\mathcal{G} + 1, 2)$, then there exists a morse meromorphic function f whose (non-removable) poles are all simple and form precisely the set $\{e_1, \dots, e_N\}$.

This discussion allows us to conclude that

Proposition 2.1. There exists a dense set of points p in M_0 such that there exists a Morse holomorphic function $f \in \mathcal{H}$ on M_0 whose (non-removable) poles are all simple and form precisely the set $\{e_1, \dots, e_N\}$ which has a critical point at p .

Proof. Let p be a point of M_0 and let u be a holomorphic function with a zero of order at least 2 at p , the existence is ensured by Lemma 2.1. Let $B(p, \eta)$ be a any small ball of radius $\eta > 0$ near p , then by Lemma 2.2, for any $\epsilon > 0$, we can approach u by a holomorphic Morse function $u_\epsilon \in \mathcal{H}_\epsilon$ whose (non-removable) poles are all simple and form precisely the set $\{e_1, \dots, e_N\}$ and which is at distance less than ϵ of u in a fixed norm on the finite dimensional space \mathcal{H} . Rouché's theorem for $\partial_z u_\epsilon$ and $\partial_z u$ (which are viewed as functions locally near p) implies that $\partial_z u_\epsilon$ has at least one zero of order exactly 1 in $B(p, \eta)$ if ϵ is chosen small enough. Thus there is a Morse function in \mathcal{H} with a critical point arbitrarily close to p . \square

Remark 2.1. In the case where the surface M has genus 0 and N ends, we have an explicit formula for the function in Proposition 2.1: indeed M_0 is conformal to $\mathbb{C} \setminus \{e_1, \dots, e_{N-1}\}$ for some $e_i \in \mathbb{C}$ - i.e. the Riemann sphere minus N points - then the function $f(z) = (z - z_0)^2 / (z - e_1)$ with $z_0 \notin \{e_1, \dots, e_{N-1}\}$ has z_0 for unique critical point in $\mathbb{C} \setminus \{e_1, \dots, e_{N-1}\}$ and it is non-degenerate.

We end this section by the following Lemmas which will be used for the amplitude of the complex geometric optics solutions but not for the phase.

Lemma 2.4. For any $p_0, p_1, \dots, p_n \in M_0$ some points of M_0 and $L \in \mathbb{N}$, then there exists a function $a(z)$ holomorphic on M_0 which vanishes to order L at all p_j for $j = 1, \dots, n$ and such that $a(p_0) \neq 0$. Moreover $a(z)$ can be chosen to have at most polynomial growth in the ends, i.e. $|a(z)| \leq C|z|^J$ for some $J \in \mathbb{N}$. The analogous statement can be made about holomorphic 1-forms.

Proof. It suffices to find on M some meromorphic function with divisor greater or equal to $D := e_1^{-J} \dots e_N^{-J} p_1^L \dots p_n^L$ but not greater or equal to Dp_0 and this is insured by Riemann-Roch theorem as long as $JN - nL \geq 2\mathcal{G}$ since then $r(D) = -\mathcal{G} + 1 + JN - nL$ and $r(Dp_0) = -\mathcal{G} + JN - nL$. \square

Lemma 2.5. Let $\{p_0, p_1, \dots, p_n\} \subset M_0$ be a set of $n+1$ disjoint points. Let $c_0, c_1, \dots, c_K \in \mathbb{C}$, $L \in \mathbb{N}$, and let z be a complex coordinate near p_0 such that $p_0 = \{z = 0\}$. Then there exists a holomorphic function f on M_0 with zeros of order at least L at each p_j , such that $f(z) = c_0 + c_1 z + \dots + c_K z^K + O(|z|^{K+1})$ in the coordinate z . Moreover f can be chosen so that there is $J \in \mathbb{N}$ such that, in the ends, $|\partial_z^\ell f(z)| = O(|z|^J)$ for all $\ell \in \mathbb{N}_0$.

Proof. The proof goes along the same lines as in Lemma 2.4. By induction on K and linear combinations, it suffices to prove it for $c_0 = \dots = c_{K-1} = 0$. As in the proof of Lemma 2.4, if J is taken large enough, there exists a function with divisor greater or equal to $D := e_1^{-J} \dots e_N^{-J} p_0^{K-1} p_1^L \dots p_n^L$ but not greater or equal to Dp_0 . Then it suffices to multiply this function by c_K times the inverse of the coefficient of z^K in its Taylor expansion at $z = 0$. \square

2.4. Laplacian on weighted spaces. Let x be a smooth positive function on M_0 , which is equal to $|z|^{-1}$ for $|z| > r_0$ in the ends $E_i \simeq \{z \in \mathbb{C}; |z| > 1\}$, where r_0 is a large fixed number. We now show that the Laplacian Δ_{g_0} on a surface with Euclidean ends has a right inverse on the weighted spaces $x^{-J}L^2(M_0)$ for $J \notin \mathbb{N}$ positive.

Lemma 2.6. For any $J > -1$ which is not an integer, there exists a continuous operator G mapping $x^{-J}L^2(M_0)$ to $x^{-J-2}L^2(M_0)$ such that $\Delta_{g_0}G = \text{Id}$.

Proof. Let $g_b := x^2 g_0$ be a metric conformal to g_0 . The metric g_b in the ends can be written $g_b = dx^2/x^2 + d\theta_{S^1}^2$ by using radial coordinates $x = |z|^{-1}, \theta = z/|z| \in S^1$, this is thus a b-metric in the sense of Melrose [19], giving the surface a geometry of surface with cylindrical ends. Let us define for $m \in \mathbb{N}_0$

$$H_b^m(M_0) := \{u \in L^2(M_0; d\text{vol}_{g_b}); (x\partial_x)^j \partial_\theta^k u \in L^2(M_0; d\text{vol}_{g_b}) \text{ for all } j+k \leq m\}.$$

The Laplacian has the form $\Delta_{g_b} = -(x\partial_x)^2 + \Delta_{S^1}$ in the ends, and the indicial roots of Δ_{g_b} in the sense of Section 5.2 of [19] are given by the complex numbers λ such that $x^{-i\lambda}\Delta_{g_b}x^{i\lambda}$ is not invertible as an operator acting on the circle S_θ^1 . Thus the indicial roots are the solutions of $\lambda^2 + k^2 = 0$ where k^2 runs over the eigenvalues of Δ_{S^1} , that is, $k \in \mathbb{Z}$. The roots are simple at $\pm ik \in i\mathbb{Z} \setminus \{0\}$ and 0 is a double root. In Theorem 5.60 of [19], Melrose proves that Δ_{g_b} is Fredholm on $x^a H_b^2(M_0)$ if and only if $-a$ is not the imaginary part of some indicial root, that is here $a \notin \mathbb{Z}$. For $J > 0$, the kernel of Δ_{g_b} on the space $x^J H_b^2(M_0)$ is clearly trivial by an energy estimate. Thus $\Delta_{g_b} : x^{-J} H_b^0(M_0) \rightarrow x^{-J} H_b^{-2}(M_0)$ is surjective for $J > 0$ and $J \notin \mathbb{Z}$, and the same then holds for $\Delta_{g_b} : x^{-J} H_b^2(M_0) \rightarrow x^{-J} H_b^0(M_0)$ by elliptic regularity.

Now we can use Proposition 5.64 of [19], which asserts, for all positive $J \notin \mathbb{Z}$, the existence of a pseudodifferential operator G_b mapping continuously $x^{-J} H_b^0(M_0)$ to $x^{-J} H_b^2(M_0)$ such that $\Delta_{g_b} G_b = \text{Id}$. Thus if we set $G = G_b x^{-2}$, we have $\Delta_{g_0} G = \text{Id}$ and G maps continuously $x^{-J+1} L^2(M_0)$ to $x^{-J-1} L^2(M_0)$ (note that $L^2(M_0) = x H_b^0(M_0)$). \square

2.5. Cauchy-Riemann Operator on Weighted Space. We begin this section with a discussion about the Fredholm properties of the operator $\bar{\partial}$ on non-compact manifolds by using the results of b-calculus. If M_0 is a surface with N Euclidean ends, then one may take the N point compactification to obtain a closed surface $M = M_0 \cup \{e_1, \dots, e_N\}$. In a holomorphic coordinate neighbourhood E_j of e_j the metric can be written in polar coordinates as $g_0 = \frac{dx^2}{x^4} + \frac{d\theta^2}{x^2}$. By extending x to be a smooth positive function on M_0 one obtains a b-metric defined by $g_b := x^2 g_0$. In this setting M_0 is a bordered manifold with x as its boundary defining function.

Let \mathcal{V}_b denote the sections of TM_0 which are tangential to ∂M_0 at the boundary and ${}^b TM_0$ be the bundle so that $\mathcal{V}_b = C^\infty(M_0, {}^b TM_0)$. Denoting its dual bundle by ${}^b T^* M_0$ one sees that near $x = 0$ the b-tangent and b-cotangent bundles are spanned by unitary (co)vectors $\{x\partial_x, \partial_\theta\}$ and $\{\frac{dx}{x}, d\theta\}$ respectively. The complexified b-cotangent bundle $\mathbb{C} {}^b T^* M_0$ has a splitting into ${}^b T_{0,1}^* M_0 \oplus {}^b T_{1,0}^* M_0$ given by the complex structure induced by g_b .

The Cauchy-Riemann operator is invariant within a conformal class of metric. However, for all $u \in C^\infty(M_0)$ it is convenient to express $\bar{\partial}u$ locally near $x = 0$ as a section of ${}^b T_{0,1}^* M_0$:

$$\bar{\partial}u = (\partial_x u + \frac{i}{x} \partial_\theta u)(dx + ixd\theta) = (x\partial_x u + i\partial_\theta u)(\frac{dx}{x} + id\theta).$$

Written in this way one sees that $\bar{\partial} \in \text{Diff}_b^1(M_0; \mathbb{C}, {}^b T_{0,1}^* M_0)$. It is also elliptic with indicial family

$$I_x(\bar{\partial}, s)u = \frac{i}{2}(-su + \partial_\theta u)(\frac{dx}{x} + id\theta)$$

which has simple roots whenever $s \in i\mathbb{Z}$ by taking $u(0, \theta) = e^{-s\theta}$. We can therefore conclude by Theorem 5.60 [19] that $\bar{\partial} : x^J H_b^m(M_0, \mathbb{C}) \rightarrow x^J H_b^{m-1}(M_0, {}^b T_{0,1}^* M_0)$ is Fredholm whenever $J \notin \mathbb{Z}$ where

$$x^J H_b^m(M_0) := \{u \in L^2(M_0; d\text{vol}_{g_b}); (x\partial_x)^j \partial_\theta^k u \in x^J L^2(M_0; d\text{vol}_{g_b}) \forall j+k \leq m\}.$$

We are now in a position to characterize the range of the $\bar{\partial}$ operator in these weighted Sobolev spaces. Indeed, if we denote by ${}^b \bar{\partial}^*$ the adjoint of $\bar{\partial}$ under the metric g_b we see that for $J \notin \mathbb{Z}$ and $m \geq 1$,

$$R_{J,m}(\bar{\partial}) = \{\omega \in x^J H_b^{m-1}(M_0, {}^b T_{0,1}^* M_0); \int_{M_0} \langle \omega, \eta \rangle_{g_b} d\text{vol}_{g_b} = 0, \forall \eta \in N_{-J, -m+1}({}^b \bar{\partial}^*)\}.$$

Here, for all $J, m \in \mathbb{R}$, $R_{J,m}(\bar{\partial})$ and $N_{J,m}({}^b \bar{\partial}^*)$ denote respectively the range and kernel of the operators $\bar{\partial}$ and ${}^b \bar{\partial}^*$ acting on their respective sections in $x^J H_b^m$. By elliptic regularity (Theorem 5.61 and (5.165) in [19]) this becomes

$$R_{J,m}(\bar{\partial}) = \{\omega \in x^J H_b^{m-1}(M_0, {}^b T_{0,1}^* M_0); \int_{M_0} \langle \omega, \eta \rangle_{g_b} d\text{vol}_{g_b} = 0, \forall \eta \in N_{-J,m}({}^b \bar{\partial}^*)\} \quad (4)$$

We look at the relationship between $N_{J,m}({}^b \bar{\partial}^*)$ and the null space of $\bar{\partial}^*$ acting on $T_{0,1}^* M_0$ when $J \in \mathbb{R} \setminus \mathbb{Z}$. If $\eta \in x^J H_b^m(M_0, {}^b T_{0,1}^* M_0)$ then it is a weighted H^m section of the bundle ${}^b T_{0,1}^* M_0$ which is a subspace of the dual space of the bundle whose smooth sections are the vector fields tangent to the boundary. Therefore, locally in the interior η has coordinate expression $\eta = u d\bar{z}$ with $u \in H^m$ and thus η is a H_{loc}^m section of $T_{0,1}^* M_0$. Near the boundary

where $x = 0$, η has the coordinate expression $\eta = u \frac{d\bar{z}}{z} = u(\frac{dx}{x} + id\theta)$ where $u \in x^J H_b^m(M_0)$. Therefore, if $\eta \in N_{J,m}(^b \bar{\partial}^*)$ then u is an antiholomorphic function satisfying $\int_{|x|<1} |x^{-J} u|^2 \frac{1}{x} dx d\theta < \infty$. Taking the Laurent series expression for u we have that u must have a zero of at least order $\lceil J \rceil$ at each end which implies that $\eta = u \frac{d\bar{z}}{z}$ has a zero of order at least $\lfloor J \rfloor$. This means that

$$\eta \in N_{J,m}(^b \bar{\partial}^*) \Rightarrow \bar{\partial}^* \eta = 0 \text{ and } \eta \in x^J L^2(M_0, d\text{vol}_{g_0}). \quad (5)$$

Furthermore, combining this discussion with standard argument about removability of singularities gives the the following Lemma and its Corollaries:

Lemma 2.7. If $\eta \in N_{J,m}(^b T_{0,1}^* M_0)$ for $J > 0$ then η extends antiholomorphically to $M = M_0 \cup \{e_1, \dots, e_N\}$ with zeroes of order at least $\lfloor J \rfloor$ at each of the ends e_j , $j = 1, \dots, N$.

Corollary 2.2. Let M_0 be a surface with $N \geq 2\mathcal{G} + 1$ ends. If $J \in \mathbb{R} \setminus \mathbb{Z}$ satisfies $J > 1$ then $N_{J,m}(^b \bar{\partial}^*)$ is trivial.

Proof. Lemma 2.7 implies that η can be extended antiholomorphically to a section of $T_{0,1}^* M$ by taking its value to be zero at e_j for $j = 1, \dots, N$. If $N \geq 2\mathcal{G} + 1$ this would force its degree (η) to be greater than or equal to $2\mathcal{G} + 1$ and thus forcing it to be the trivial section. \square

Corollary 2.3. Let M_0 be a surface with $N \geq 2\mathcal{G} + 1$ ends. If $J \in \mathbb{R} \setminus \mathbb{Z}$ satisfies $2 > J > 1$ then $R_{-J,m}(\bar{\partial}) = x^{-J} H_b^{m-1}(M_0; ^b T_{0,1}^* M_0)$. Furthermore, there exists a bounded operator

$$\bar{\partial}_J^{-1} : x^{-J} H_b^{m-1}(M_0; ^b T_{0,1}^* M_0) \rightarrow x^{-J} H_b^m(M_0; \mathbb{C})$$

satisfying $\bar{\partial} \bar{\partial}_J^{-1} = Id$, $\bar{\partial}_J^{-1} \bar{\partial} = Id - \Pi$ where

$$\Pi : x^{-J} L_b^2(M_0; \mathbb{C}) \rightarrow N_{-J,m}(\bar{\partial}) \subset x^{-J} H_b^m(M_0; \mathbb{C})$$

is the orthogonal projection on $x^{-J} L_b^2$ with respect to the inner product

$$\langle u, v \rangle_J := \int_{M_0} x^{2J} \bar{u} v d\text{vol}_{g_b}.$$

Proof. Combine Corollary 2.2 and (4) we have that the operator

$$\bar{\partial} : x^{-J} H_b^m(M_0, \mathbb{C}) \rightarrow x^{-J} H_b^{m-1}(M_0; ^b T_{0,1}^* M_0)$$

is surjective. Theorem 5.60 [19] also states that this operator is Fredholm so there exists a generalized inverse

$$\bar{\partial}_J^{-1} : x^{-J} H_b^{m-1}(M_0; ^b T_{0,1}^* M_0) \rightarrow x^{-J} H_b^m(M_0; \mathbb{C})$$

satisfying $\bar{\partial} \bar{\partial}_J^{-1} = Id$, $\bar{\partial}_J^{-1} \bar{\partial} = Id - \Pi$ where

$$\Pi : x^{-J} H_b^m(M_0; \mathbb{C}) \rightarrow N_{-J,m}(\bar{\partial}) \subset x^{-J} H_b^m(M_0; \mathbb{C})$$

is the orthogonal projection described in the statement of the Corollary. \square

In the case when η is compactly supported we can easily work out the expression for the kernel of $\bar{\partial}_J^{-1}$ by using the existing machinery. Indeed, if $K = \text{supp}(\eta)$ is contained in the interior of M_0 then let $\{\chi_j\}_{j=1}^l$ be a partition of unity by $C_0^\infty(M_0)$ functions for some open cover of K by holomorphic coordinate neighbourhoods $\{U_j\}_{j=1}^l$ with $U_j \subset\subset M_0$. Let χ'_j be compactly supported smooth function in U_j which is equal to 1 in a neighbourhood of the support of χ_j . Define

$$T\eta := \sum_j \chi'_j \int_{U_j} \frac{\chi_j(z') f_j(z')}{z' - z} dz' \wedge d\bar{z}' \quad (6)$$

where $f_j d\bar{z}$ is the coordinate expression of η in U_j . One immediately gets that

$$\bar{\partial} T\eta = \eta + \left(\sum_j \omega_j \int_{U_j} \kappa_j(z', z) f_j(z') dz' \wedge d\bar{z}' \right)$$

where κ_j are smooth compactly supported functions in $U_j \times U_j$ and ω_j are smooth sections of $T_{0,1}^* M_0$ compactly supported in U_j . Hitting both sides with $\bar{\partial}_J^{-1}$ for $2 > J > 1$ we get from Corollary 2.3

$$\bar{\partial}_J^{-1} \eta = (Id + \Pi) T\eta - \bar{\partial}_J^{-1} \sum_j \omega_j \int_{U_j} \kappa_j(z', z) f_j(z') dz' \wedge d\bar{z}' \quad (7)$$

The expression (7) combined with the explicit formula for the T operator given by (6) allows us to prove the following

Lemma 2.8. If ψ is a Morse function on M_0 and η is a compactly supported smooth section of $T_{0,1}^* M_0$ then for $2 > J > 1$ we have

$$\|x^J \bar{\partial}_J^{-1} (e^{2i\psi/h} \eta)\|_{L_b^2(M_0)} \leq Ch^{\frac{1}{2}+\epsilon}.$$

Here the constant depends on η and the size of its support.

Proof. Using the expression (7) and replacing η by $e^{i2\psi/h} \eta$ we see that $\bar{\partial}_J^{-1} e^{-2i\psi/h} \eta$ can be bounded by two separate terms. The operator $\bar{\partial}_J^{-1}$ is bounded from $x^{-J} H_b^{m-1}(M_0, {}^b T_{0,1}^* M_0)$ to $x^{-J} H_b^m(M_0)$ and the last term is the composition of this operator with a finite sum of integrals against smooth compactly supported kernels. Therefore, the last term can be treated with stationary phase to show that its $x^{-J} L_b^2$ norm is of order h .

The first term can be estimated by using the explicit expression of the kernel given in (6) and the fact that $(Id + \Pi)$ is bounded on $x^{-J} L_b^2(M_0)$. Since each of χ'_j are compactly supported and we are summing over finitely many terms in (6), repeating the same argument as Lemma 2.2 of [10] would yield that $\|x^J T e^{2i\psi/h} \eta\|_{L_b^2} \leq C(\text{supp}(\eta)) h^{\frac{1}{2}+\epsilon} \|\eta\|_{W^{1,p}}$ and the proof is complete by the boundedness of $(Id - \Pi)$. \square

2.6. Construction of Conjugation Factor. If A is a section of $T_{0,1}^* M_0$ then the operator $\bar{\partial} + iA$ is a Cauchy-Riemann operator acting on the trivial complex line bundle over M_0 . By [17] there exists a unique holomorphic structure which is compatible with this Cauchy-Riemann operator and it is trivialized by a non-vanishing section of the bundle. It is useful to construct

an explicit form of this trivialization. In particular, if α is a differentiable function satisfying $\bar{\partial}\alpha = A$, then one has $\bar{\partial} + iA = e^{-i\alpha}\bar{\partial}e^{i\alpha}$. It is important in this article to understand the asymptotic of the trivialization near the ends. This motivates us to consider the following construction:

Lemma 2.9. Let $\eta \in e^{-\gamma|z|}H^{3+\epsilon_0}(\mathbb{C})$ for all $\gamma > 0$. We have the following expansion for the parametrix of the Cauchy-Riemann operator

$$\bar{\partial}^{-1}\eta(z) := \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\eta(\zeta)}{z - \zeta} d\zeta \wedge d\bar{\zeta} = \frac{c_{-1}}{z} + \cdots + \frac{c_{-k}}{z^k} + R_k(z), \quad (8)$$

where $R_k(z) := z^{-k}\bar{\partial}^{-1}(z^k\eta)$, with $|\partial_z^j R_k(z)| \leq C_{j,k}|z|^{-1}\|e^{\gamma|\zeta|}\eta(\zeta)\|_{H^{1+j+\epsilon_0}}$, for all $j = 0, 1, 2$.

Proof. One sees easily that

$$\int_{\mathbb{C}} \frac{\eta(\zeta)}{z - \zeta} d\zeta \wedge d\bar{\zeta} = \frac{1}{z} \int \eta d\zeta \wedge d\bar{\zeta} + \frac{1}{z} \int_{\mathbb{C}} \frac{\zeta\eta(\zeta)}{z - \zeta} d\zeta \wedge d\bar{\zeta}$$

so using the fact that η decays super-exponentially we can iterate this relation get the expansion (8).

To get the estimate on the remainder we observe again that since η decays super-exponentially it suffices to do this for $k = 0$. This can be done for $j = 0$, by first observing that $\eta \in e^{-\gamma|z|}L^\infty(\mathbb{C})$, by the Sobolev embedding Theorem, if $\eta \in e^{-\gamma|\zeta|}H^{1+\epsilon_0}$ and then splitting the integral and estimating as follows

$$\int_{|\zeta| \leq \frac{|z|}{2}} \frac{\eta(z - \zeta)}{\zeta} d\zeta \wedge d\bar{\zeta} + \int_{|\zeta| \geq \frac{|z|}{2}} \frac{\eta(z - \zeta)}{\zeta} d\zeta \wedge d\bar{\zeta} \leq C(|z|e^{-\gamma|z|} + |z|^{-1}).$$

For $j = 1$ the estimate can be done by observing that $[\partial_z, \bar{\partial}^{-1}]\eta$ is an entire function and furthermore $\partial_z \bar{\partial}^{-1}\eta \in L^2$ by Calderón-Zygmund while $\bar{\partial}^{-1}\partial_z\eta = O(|z|^{-1})$ by the $j = 0$ case. We conclude then that $[\partial_z, \bar{\partial}^{-1}]\eta$ is an entire function which is in $\langle |z| \rangle^\epsilon L^2(\mathbb{C})$ for all $\epsilon > 0$. This forces $[\partial_z, \bar{\partial}^{-1}]\eta = 0$ which means $\partial_z \bar{\partial}^{-1}\eta = \bar{\partial}^{-1}\partial_z\eta = O(|z|^{-1})$ by the fact that $\eta \in e^{-\gamma|z|}H^l(\mathbb{C})$ and using the $j = 0$ estimate. This argument can be made as well for $j = 2$ (in fact as many times as the differentiability of η allows) and the proof is complete. \square

Lemma 2.10. Let $\eta \in H^1(M_0; T_{0,1}^*M_0)$ be a compactly supported 1-form. Then there exists solutions to the equation $\bar{\partial}\alpha = \eta$ which has uniformly convergent power series expansion

$$\alpha = c_1 z + c_0 + \sum_{j=1}^{\infty} c_j z^{-j}$$

for $|z|$ large.

Proof. Denoting by $x := |z|^{-1}$ it is clear that since η is compactly supported it belongs to $x^{-J}H_b^1(M_0; {}^bT_{0,1}^*M_0)$ for $2 > J > 1$ and by Corollary 2.3 we see that there exists a unique $\alpha \in x^{-J}H_b^2(M_0; \mathbb{C})$ solving $\bar{\partial}\alpha = \eta$. As η is compactly supported, α is actually holomorphic for large $|z|$ in the ends E_j and has a Laurent series expansion $\alpha = \sum_{j=-\infty}^{\infty} c_j z^j$ which converges uniformly

in the annulus $R_1 < |z| < R_2$ for $0 < R_1 < R_2$ large enough. By the fact that $\alpha \in x^{-J}H_b^2(M_0; \mathbb{C})$ with $2 > J > 1$ this forces $c_j = 0$ for $j \geq 2$. \square

Combining Lemmas 2.9 and 2.10 we obtain the following

Proposition 2.2. Let $\eta \in e^{-\gamma/x}H^l(M_0; T_{0,1}^*M_0)$ for all $\gamma > 0$. There exists a function $\alpha \in x^{-J}H^{l+1}(M_0; \mathbb{C})$, $2 > J > 1$ solving $\bar{\partial}\alpha = \eta$ which for any k has expansion

$$|\partial_z^j(\alpha - (c_1z + c_0 + \cdots + c_{-k}z^{-k}))| \leq C_{j,k}|z|^{-(k+1)}$$

near the ends when $|z| \rightarrow \infty$ for $j = 0, 1, 2$.

3. CARLEMAN ESTIMATES AND SOLVABILITY

In this section, we prove a Carleman estimate using harmonic weights with non-degenerate critical points. Our starting point is the estimates in [11], which are used to obtain an estimate for negative order Sobolev spaces. Duality will then allows us to prove a H_{scl}^1 solvability result for the magnetic operator, that will later needed in constructing complex geometric optics solutions. We remind the reader that we use Δ_g to denote the positive Laplacian.

We first consider a Morse holomorphic function $\Phi \in \mathcal{H}$ obtained from Proposition 2.1 with the condition that Φ has linear growth in the ends. We will write

$$\Phi := \varphi + i\psi, \quad \text{where } \varphi := \text{Re}(\Phi), \psi := \text{Im}(\Phi). \quad (9)$$

The Carleman weight will consist of the harmonic function $\varphi = \text{Re}(\Phi)$. We let x be a positive smooth function on M_0 such that $x = |z|^{-1}$ in the complex charts $\{z \in \mathbb{C}; |z| > 1\} \simeq E_j$ covering the end E_j . We will assume without loss of generality that Φ (and therefore φ) does not have critical points in $\overline{E_j}$.

We modify our weight using a function φ_0 . Let $\delta \in (0, 1)$ be small and let us take $\varphi_0 \in x^{-\beta}L^2(M_0)$ a solution of $\Delta_{g_0}\varphi_0 = x^{2-\delta}$, a solution exists by Proposition 2.6 if $\beta > 1 + \delta$. Actually, by using Proposition 5.61 of [19], if we choose $\beta < 2$, then it is easy to see that φ_0 is smooth on M_0 and has polyhomogeneous expansion as $|z| \rightarrow \infty$, with leading asymptotic in the end E_i given by $\varphi_0 = -x^{-\delta}/\delta^2 + c_i \log(x) + d_i + O(x)$ for some c_i, d_i which are smooth functions in S^1 .

We will modify our weight function one step further to allow more generality. We assume that α is as in Proposition 2.2, with $\bar{\partial}\alpha = \eta \in e^{-\gamma/x}H^{3+\epsilon_0}(M_0)$. In particular that α has a leading asymptotics in the end E_j given by

$$|\partial_z^j(\alpha - (c_1z + c_0 + \cdots + c_{-k}z^{-k}))| \leq C_{j,k}|z|^{-(k+1)}$$

near the ends when $|z| \rightarrow \infty$ for $j = 0, 1, 2$. For $\epsilon > 0$ small, we define the convexified weight

$$\varphi_\epsilon := \varphi + h\text{Re}(i\alpha) - \frac{h}{\epsilon}\varphi_0. \quad (10)$$

It follows that φ_ϵ has an expansion at infinity of the form

$$\varphi_\epsilon(z) = \gamma \cdot z + \frac{h}{\epsilon} \frac{r^\delta}{\delta^2} + c_1 \log(r) + c_2 + c_3 r^{-1} + O(r^{-2}),$$

where $r = |z|$, $\omega = z/r$, $\gamma = (\gamma_1 + h\gamma'_1, \gamma_2 + h\gamma'_2) \in \mathbb{R}^2$, $z = (z_1, z_2) \in \mathbb{R}^2$, and c_i are some smooth functions on S^1 depending on h . Moreover we have that

$$\begin{aligned} d\varphi_\epsilon &= \gamma_1 dz_1 + \gamma_2 dz_2 + O(r^{-1+\delta}), \\ \partial_z^\kappa \partial_{\bar{z}}^\mu \varphi_\epsilon(z) &= O(r^{-2+\delta}) \quad \text{for all } \kappa + \mu \geq 2. \end{aligned} \tag{11}$$

The following estimate was proved in [34] with $\gamma'_1 = \gamma'_2 = 0$ but as they are lower order terms in the phase and the domain one considers is compact, the same proof holds in the slightly more general case of φ_ϵ . See Proposition 3.1 in [34] for details.

Proposition 3.1. Let $K \subset M_0$ be compact and the φ_ϵ the previously defined weight. Then for $u \in C_0^\infty(K)$, we have

$$\frac{Ch}{\epsilon} \left(\sqrt{h} \|u\|_{L^2} + \|d\varphi_\epsilon u\|_{L^2} + \|hdu\|_{H_{scl}^{-1}} \right) \leq \|e^{\varphi_\epsilon/h} h^2 \Delta_g e^{-\varphi_\epsilon/h} u\|_{H_{scl}^{-1}},$$

where C depends on K but not on h and ϵ .

We will use semiclassical pseudodifferential calculus in the following proofs. A function $a \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ is in the semiclassical symbol class $\mathcal{S}^k(\langle \xi \rangle^m)$, if

$$|\partial_x^\alpha \partial_\xi^\beta a(y, \xi; h)| \leq C_{\alpha\beta} h^{-k} \langle \xi \rangle^m, \tag{12}$$

where $C_{\alpha\beta}$ is independent of the parameter h , see [28] and [40]. We will use the abbreviation $\mathcal{S}^m := \mathcal{S}^0(\langle \xi \rangle^m)$.

We will need to prove an analogue of Proposition 3.1 for the ends. Combining these will give us a global estimate in the semiclassical H^{-1} -norm. We begin by proving the following weighted L^2 -estimate, which is essentially the same as Proposition 3.1 in [11], apart from the more general weight function used here. We give the proof here as a convenience to the reader.

Proposition 3.2. Let $\delta \in (0, 1)$, and φ_ϵ as above, then there exists $C > 0$ such that for all $\epsilon \gg h > 0$ small enough, and all $u \in C_0^\infty(E_j)$

$$h^2 \|e^{\varphi_\epsilon/h} (\Delta - \lambda^2) e^{-\varphi_\epsilon/h} u\|_{L^2}^2 \geq \frac{C}{\epsilon} (\|x^{1-\frac{\delta}{2}} u\|_{L^2}^2 + h^2 \|x^{1-\frac{\delta}{2}} du\|_{L^2}^2).$$

Proof. The metric g_0 can be extended to \mathbb{R}^2 to be the Euclidean metric and we shall denote by Δ the flat positive Laplacian on \mathbb{R}^2 . Let us write $P := \Delta_{g_0} - \lambda^2$, then the operator $P_h := h^2 e^{\varphi_\epsilon/h} P e^{-\varphi_\epsilon/h}$ is given by

$$P_h = h^2 \Delta - |d\varphi_\epsilon|^2 + 2h \nabla \varphi_\epsilon \cdot \nabla - h \Delta \varphi_\epsilon - h^2 \lambda^2,$$

is a semiclassical operator with a semiclassical full Weyl symbol

$$\sigma_w(P_h) := |\xi|^2 - |d\varphi_\epsilon|^2 - h^2 \lambda^2 + 2i \langle d\varphi_\epsilon, \xi \rangle = a + ib \in \mathcal{S}^2.$$

We can define $A := (P_h + P_h^*)/2 = h^2 \Delta - |d\varphi_\epsilon|^2 - h^2 \lambda^2$ and $B := (P_h - P_h^*)/2i = -2ih \nabla \varphi_\epsilon \cdot \nabla + ih \Delta \varphi_\epsilon$ which have respective semiclassical full symbols a and b , i.e. $A = \text{Op}_w(a)$ and $B = \text{Op}_w(b)$ for the Weyl quantization.

Notice that A, B are symmetric operators, thus for all $u \in C_0^\infty(E_i)$

$$\|(A + iB)u\|^2 = \langle (A^2 + B^2 + i[A, B])u, u \rangle. \quad (13)$$

It is easy to check that the operator $ih^{-1}[A, B]$ is a semiclassical differential operator in \mathcal{S}^2 with full semiclassical symbol

$$\{a, b\}(\xi) = 4(D^2\varphi_\epsilon(d\varphi_\epsilon, d\varphi_\epsilon) + D^2\varphi_\epsilon(\xi, \xi)) \quad (14)$$

Let us now decompose the Hessian of φ_ϵ in the basis $(d\varphi_\epsilon, \theta)$ where θ is a covector orthogonal to $d\varphi_\epsilon$ and of norm $|d\varphi_\epsilon|$. This yields coordinates $\xi = \xi_0 d\varphi_\epsilon + \xi_1 \theta$ and there exist smooth functions M, N, K so that

$$D^2\varphi_\epsilon(\xi, \xi) = |d\varphi_\epsilon|^2(M\xi_0^2 + N\xi_1^2 + 2K\xi_0\xi_1).$$

The asymptotics in (11) imply that $M, N, K \in r^{-2+\delta}L^\infty(E_i)$. Then one can write

$$\begin{aligned} \{a, b\} &= 4|d\varphi_\epsilon|^2(M + M\xi_0^2 + N\xi_1^2 + 2K\xi_0\xi_1) \\ &= 4(N(a + h^2\lambda^2) + ((M - N)\xi_0 + 2K\xi_1)b/2 + (N + M)|d\varphi_\epsilon|^2) \end{aligned}$$

and since

$$M + N = \text{Tr}(D^2\varphi_\epsilon) = -\Delta\varphi_\epsilon = h\Delta\text{Re}(i\alpha) + h\Delta\varphi_0/\epsilon = \frac{h}{\epsilon}x^{2-\delta} + h\text{Re}(i\Delta\alpha)$$

with $\Delta\alpha = \bar{\partial}^*\eta \in e^{-\gamma|z|}H^{2+\epsilon_0}(\mathbb{R}^2) \subset e^{-\gamma|z|}W^{1,\infty}(\mathbb{R}^2)$ we obtain

$$\begin{aligned} \{a, b\} &= 4|d\varphi_\epsilon|^2(c(z)(a + h^2\lambda^2) + \ell(z, \xi)b + \frac{h}{\epsilon}r^{-2+\delta} + h\Delta\text{Re}(i\alpha)), \\ c(z) &= \frac{N}{|d\varphi_\epsilon|^2}, \quad \ell(z, \xi) = \frac{(M - N)\xi_0 + 2K\xi_1}{2|d\varphi_\epsilon|^2}. \end{aligned} \quad (15)$$

Now, we take a smooth extension of $|d\varphi_\epsilon|^2, a(z, \xi), \ell(z, \xi), \alpha(z)$ and r to $z \in \mathbb{R}^2$, this can be done for instance by extending r as a smooth positive function on \mathbb{R}^2 and then extending $d\varphi$ and $d\varphi_0$ to smooth non vanishing 1-forms on \mathbb{R}^2 (not necessarily exact) so that $|d\varphi_\epsilon|^2$ is smooth positive (for small h) and polynomial in h and a, ℓ are of the same form as in $\{|z| > 1\}$. Let us define the symbol and quantized differential operator on \mathbb{R}^2

$$e := 4|d\varphi_\epsilon|^2(c(z)(a + h^2\lambda^2) + \ell(z, \xi)b), \quad E := \text{Op}_w(e)$$

and write

$$\begin{aligned} ih^{-1}r^{1-\frac{\delta}{2}}[A, B]r^{1-\frac{\delta}{2}} &= hF + r^{1-\frac{\delta}{2}}Er^{1-\frac{\delta}{2}} - \frac{h}{\epsilon}(A^2 + B^2), \\ \text{with } F &:= h^{-1}r^{1-\frac{\delta}{2}}(ih^{-1}[A, B] - E)r^{1-\frac{\delta}{2}} + \frac{1}{\epsilon}(A^2 + B^2). \end{aligned} \quad (16)$$

We deduce from (14) and (15) the following

Lemma 3.1. The operator F is a semiclassical differential operator in the class \mathcal{S}^4 with semiclassical principal symbol

$$\sigma_w(F)(\xi) = 4|d\varphi|^2\left(\frac{1}{\epsilon} + r^{2-\delta}\text{Re}(i\Delta\alpha)\right) + \frac{1}{\epsilon}(|\xi|^2 - |d\varphi|^2)^2 + \frac{4}{\epsilon}(\langle \xi, d\varphi \rangle)^2.$$

By the semiclassical Gårding estimate, we obtain the

Corollary 3.1. The operator F of Lemma 3.1 is such that there is a constant C so that

$$\langle Fu, u \rangle \geq \frac{C}{\epsilon} (\|u\|_{L^2}^2 + h^2 \|du\|_{L^2}^2).$$

Proof. It suffices to use that when $\epsilon > 0$ is chosen to be small enough, $\sigma_w(F)(\xi) \geq \frac{C'}{\epsilon}(1 + |\xi|^4)$ for some $C' > 0$ and use the semiclassical Gårding estimate. The symbol estimate comes from the fact that $|d\varphi|$ is bounded away from 0 and $\Delta \text{Re}(i\alpha)$ decays superexponentially. \square

So by writing $\langle i[A, B]u, u \rangle = \langle ir^{1-\frac{\delta}{2}}[A, B]r^{1-\frac{\delta}{2}}r^{-1+\frac{\delta}{2}}u, r^{-1+\frac{\delta}{2}}u \rangle$ in (13) and using (16) and Corollary 3.1, we obtain that there exists $C > 0$ such that for all $u \in C_0^\infty(E_i)$

$$\begin{aligned} \|P_h u\|_{L^2}^2 &\geq \langle (A^2 + B^2)u, u \rangle + \frac{Ch^2}{\epsilon} (\|r^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + h^2 \|r^{-1+\frac{\delta}{2}}du\|_{L^2}^2) \\ &\quad + h \langle Eu, u \rangle - \frac{h^2}{\epsilon} (\|A(r^{-1+\frac{\delta}{2}}u)\|_{L^2}^2 + \|B(r^{-1+\frac{\delta}{2}}u)\|_{L^2}^2). \end{aligned} \quad (17)$$

We observe that $h^{-1}[A, r^{-1+\frac{\delta}{2}}]r^{1+\frac{\delta}{2}} \in \mathcal{S}^1$ and $h^{-1}[B, r^{-1+\frac{\delta}{2}}]r^{1+\frac{\delta}{2}} \in h\mathcal{S}^0$, and thus

$$\begin{aligned} \|A(r^{-1+\frac{\delta}{2}}u)\|_{L^2}^2 + \|B(r^{-1+\frac{\delta}{2}}u)\|_{L^2}^2 &\leq C' (\|Au\|_{L^2}^2 + \|Bu\|_{L^2}^2 \\ &\quad + h^2 \|r^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + h^4 \|r^{-1+\frac{\delta}{2}}du\|_{L^2}^2) \end{aligned}$$

for some $C' > 0$. Taking h small, this implies with (17) that there exists a new constant $C > 0$ such that

$$\begin{aligned} \|P_h u\|_{L^2}^2 &\geq \frac{Ch^2}{\epsilon} (\|r^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + h^2 \|r^{-1+\frac{\delta}{2}}du\|_{L^2}^2) \\ &\quad + \frac{1}{2} \langle (A^2 + B^2)u, u \rangle + h \langle Eu, u \rangle. \end{aligned} \quad (18)$$

It remains to deal with $h \langle Eu, u \rangle$: we first write $E = 4|d\varphi_\epsilon|^2(c(z)(A + h^2\lambda^2) + \text{Op}_w(\ell)B) + hr^{-1+\frac{\delta}{2}}Sr^{-1+\frac{\delta}{2}}$ where S is a semiclassical differential operator in the class \mathcal{S}^1 by the decay estimates on $c(z), \ell(z, \xi)$ as $z \rightarrow \infty$, then by Cauchy-Schwartz (and with $L := \text{Op}_w(\ell)$)

$$\begin{aligned} |\langle hEu, u \rangle| &\leq Ch(\|Au\|_{L^2} + h^2 \|r^{-1+\frac{\delta}{2}}u\|_{L^2} + h \|Sr^{-1+\frac{\delta}{2}}u\|_{L^2}) \|r^{-1+\frac{\delta}{2}}u\|_{L^2} \\ &\quad + Ch \|Bu\|_{L^2} \|Lu\|_{L^2} \\ &\leq \frac{1}{4} \|Au\|_{L^2}^2 + h^2 \|Sr^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + Ch^2 \|r^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + \frac{1}{4} \|Bu\|_{L^2}^2 \\ &\quad + Ch^2 \|Lu\|_{L^2}^2 \end{aligned}$$

where C is a constant independent of h, ϵ but may change from line to line. Now we observe that $Lr^{1-\frac{\delta}{2}}$ and S are in \mathcal{S}^1 and thus

$$\|Sr^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + \|Lu\|_{L^2}^2 \leq C(\|r^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + h^2 \|r^{-1+\frac{\delta}{2}}du\|_{L^2}^2),$$

which by (18) implies that there exists $C > 0$ such that for all $\epsilon \gg h > 0$ with ϵ small enough

$$\|P_h u\|_{L^2}^2 \geq \frac{Ch^2}{\epsilon} (\|r^{-1+\frac{\delta}{2}}u\|_{L^2}^2 + h^2 \|r^{-1+\frac{\delta}{2}}du\|_{L^2}^2)$$

for all $u \in C_0^\infty(E_i)$. The proof is complete. \square

In the following proofs we need some additional facts about the semiclassical calculus. Firstly recall that a symbol $a \in \mathcal{S}^m$ corresponds in the so called classical quantization to an operator $\text{Op}_h(a) = a(y, hD)$ defined by

$$\text{Op}_h(a)f(y) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{iy \cdot \xi} a(y, h\xi; h) \hat{f}(\xi) d\xi.$$

We use $\sigma(A)$ to denote the symbol corresponding to a semiclassical operator A .

Moreover we need a formula for the commutator of two semiclassical operators with symbols $a \in \mathcal{S}^m$ and $b \in \mathcal{S}^{m'}$. We have that $\sigma([\text{Op}_h(a), \text{Op}_h(b)]) \in \mathcal{S}^{m+m'}$ and

$$\sigma([\text{Op}_h(a), \text{Op}_h(b)]) = \frac{h}{i} (\nabla_\xi a \cdot \nabla_y b - \nabla_y a \cdot \nabla_\xi b) + h^2 \mathcal{S}^{m+m'}, \quad (19)$$

See [28]. We shall moreover utilize the following Proposition from [28]. It is convenient to formulate this using the weighted semiclassical spaces $H_{\delta, scl}^s$, defined by the norm $\|f\|_{H_{\delta, scl}^s} := \|\langle hD \rangle^s \langle y \rangle^\delta f\|_{L^2}$.

Proposition 3.3. Let $a \in \mathcal{S}^0$ and $\delta_0 \geq 0$. Then $\text{Op}_h(a)$ is bounded $H_{\delta, scl}^s(\mathbb{R}^n) \rightarrow H_{\delta, scl}^s(\mathbb{R}^n)$ for any $s, \delta \in \mathbb{R}$, and there is a constant C with $\|\text{Op}_h(a)\|_{H_{\delta, scl}^s \rightarrow H_{\delta, scl}^s} \leq C$ whenever $|s| \leq s_0$, $|\delta| \leq \delta_0$, and $0 < h \leq h_0$.

We now prove a weighted version of Proposition 3.1 that holds in the ends. This is done by shifting the estimate of Proposition 3.2.

Proposition 3.4. Let $\delta \in (0, 1)$, and φ_ϵ as above, then there exists $C > 0$ such that for all $\epsilon \gg h > 0$ small enough, and all $u \in C_0^\infty(E_j)$,

$$\frac{C}{\epsilon} \|x^{1-\frac{\delta}{2}} u\|_{L^2} \leq h \|e^{\varphi_\epsilon/h} (\Delta - \lambda^2) e^{-\varphi_\epsilon/h} u\|_{H_{scl}^{-1}}.$$

Proof. We will employ the same notations as in the proof of Proposition 3.2. In particular $r = x^{-1}$ and $P_h := e^{\varphi_\epsilon/h} h^2 (\Delta - \lambda^2) e^{-\varphi_\epsilon/h}$.

Let $\chi \in C^\infty(\mathbb{R}^2)$, be such that $\chi(y) = 1$, in $\mathbb{R}^2 \setminus \mathbb{D}_1$ and $\chi(y) = 0$, near $\overline{\mathbb{D}}_{1/2}$. Now consider the function $\chi \langle hD \rangle^{-1} u$, where $u \in C_0^\infty(\mathbb{R}^2 \setminus \mathbb{D}_1)$. It is straight forward to see, using a density argument that Proposition 3.2 applies to functions in the Schwartz class, so that we may apply it to $\chi \langle hD \rangle^{-1} u$ and get that

$$\begin{aligned} \|r^{\frac{\delta}{2}-1} \chi \langle hD \rangle^{-1} u\|_{L^2} + h \|r^{\frac{\delta}{2}-1} d(\chi \langle hD \rangle^{-1} u)\|_{L^2} \\ \leq C \epsilon \|P_h(\chi \langle hD \rangle^{-1} u)\|_{L^2}. \end{aligned} \quad (20)$$

Let $\theta = \delta/2 - 1$, so that $r^{\frac{\delta}{2}-1} = r^\theta$. We want to estimate the left hand side from below, by $\|r^{\frac{\delta}{2}-1} u\|_{L^2}$. This is equivalent to estimating it from below by $\|u\|_{L_\theta^2}$. We start by writing

$$\begin{aligned} \|r^\theta \chi \langle hD \rangle^{-1} u\|_{L^2} + h \|r^\theta d(\chi \langle hD \rangle^{-1} u)\|_{L^2} \\ \geq \|\chi \langle hD \rangle^{-1} u\|_{H_{\theta, scl}^1} - h \|[d, r^\theta](\chi \langle hD \rangle^{-1} u)\|_{L^2}, \end{aligned}$$

We can absorb the commutator term by the first term on the left hand side, when h is small, since $[d, r^\theta] = \theta r^{\theta-1}$. It is hence enough to estimate the term containing the weighted Sobolev norm from below. We have that

$$\|\chi \langle hD \rangle^{-1} u\|_{H_{\theta, scl}^1} \geq \|\langle hD \rangle^{-1} (\chi u)\|_{H_{\theta, scl}^1} - \|[\chi, \langle hD \rangle^{-1}] u\|_{H_{\theta, scl}^1} \quad (21)$$

By expression (19) for the symbol of a commutator we have that

$$\sigma([\chi, \langle hD \rangle^{-1}]) = -\frac{h}{i} \nabla_y \chi \nabla_\xi \langle \xi \rangle^{-1} + h^2 \mathcal{S}^{-1}.$$

It follows then from Proposition 3.3 that

$$\|[\chi, \langle hD \rangle^{-1}] u\|_{H_{\theta, scl}^1} \leq Ch \|u\|_{L_\theta^2}. \quad (22)$$

Next we estimate the middle term in (21), as follows

$$\begin{aligned} \|\langle hD \rangle^{-1} u\|_{H_{\theta, scl}^1} &= \|\langle r \rangle^\theta \langle hD \rangle^{-1} u\|_{H_{scl}^1} \\ &\geq C \|\langle hD \rangle^{-1} (\langle r \rangle^\theta u)\|_{H_{scl}^1} - \|[\langle r \rangle^\theta, \langle hD \rangle^{-1}] u\|_{H_{scl}^1} \\ &\geq C \|u\|_{L_\theta^2} - \|[\langle r \rangle^\theta, \langle hD \rangle^{-1}] u\|_{H_{scl}^1} \end{aligned}$$

The commutator can be estimated by Lemma 3.2

$$\|[\langle r \rangle^\theta, \langle hD \rangle^{-1}] u\|_{H_{scl}^1} \leq Ch \|\langle r \rangle^{\theta-1} u\|_{L^2} \leq Ch \|u\|_{L_\theta^2}.$$

We can hence absorb the commutator by the first term, when h is small and get

$$\|\langle hD \rangle^{-1} u\|_{H_{\theta, scl}^1} \geq C \|u\|_{L_\theta^2}.$$

It follows from (21) using the above estimate and (22) that

$$\begin{aligned} \|\chi \langle hD \rangle^{-1} u\|_{H_{\theta, scl}^1} &\geq C \|u\|_{L_\theta^2} - Ch \|u\|_{L_\theta^2} \\ &\geq C \|u\|_{L_\theta^2}, \end{aligned}$$

when h is small. We can thus estimate the left hand side of (20) from below as follows

$$\frac{C}{\epsilon} \|r^{\frac{\delta}{2}-1} u\|_{L^2} \leq \|P_h(\chi \langle hD \rangle^{-1} u)\|_{L^2}. \quad (23)$$

Splitting the right hand side of (23) using the basic properties of commutators, gives that

$$\begin{aligned} \|P_h(\chi \langle hD \rangle^{-1} u)\|_{L^2} &\leq \|\chi \langle hD \rangle^{-1} P_h u\|_{L^2} \\ &\quad + \|\chi [P_h, \langle hD \rangle^{-1}] u\|_{L^2} \\ &\quad + h^2 \|e^{\varphi_\epsilon/h} [\Delta - \lambda^2, \chi] (e^{-\varphi_\epsilon/h} \langle hD \rangle^{-1} u)\|_{L^2}. \end{aligned} \quad (24)$$

To obtain the estimate in the statement of the Proposition, we need to show that the second and third term can be absorbed by the left hand side of (23), when ϵ is chosen small enough. This can be done if we can bound these terms in the weighted L^2 -norm.

Writing out the commutator in the third term yields

$$\begin{aligned} h^2 \|e^{\varphi_\epsilon/h} [\Delta, \chi] (e^{-\varphi_\epsilon/h} \langle hD \rangle^{-1} u)\|_{L^2} &\leq Ch^2 (\|\Delta \chi \langle hD \rangle^{-1} u\|_{L^2} \\ &\quad + \|e^{\varphi_\epsilon/h} \nabla \chi \cdot \nabla e^{-\varphi_\epsilon/h} \langle hD \rangle^{-1} u\|_{L^2} \\ &\quad + \|\nabla \chi \nabla (\langle hD \rangle^{-1} u)\|_{L^2}). \end{aligned}$$

We now find bounds for the terms on the right hand side in the weighted L^2 -norm. For the last term we note that $\sigma(h\nabla \langle hD \rangle^{-1}) \in \mathcal{S}^0$. By Proposition 3.3 we know that $h\nabla \langle hD \rangle^{-1}: L_\theta^2 \rightarrow L_\theta^2$ is continuous and hence that

$$h^2 \|\nabla \chi \nabla (\langle hD \rangle^{-1} u)\|_{L^2} \leq Ch \|h\nabla (\langle hD \rangle^{-1} u)\|_{L_\theta^2} \leq Ch \|u\|_{L_\theta^2}.$$

For the two remaining terms, we have that $\sigma(\langle hD \rangle^{-1}) \in \mathcal{S}^{-1}$. By Proposition 3.3 we know that $\langle hD \rangle^{-1}: L_\theta^2 \rightarrow L_\theta^2$ is continuous and thus we have in the same way that

$$h^2 (\|\Delta \chi \langle hD \rangle^{-1} u\|_{L^2} + \|e^{\varphi_\epsilon/h} \nabla \chi \cdot \nabla e^{-\varphi_\epsilon/h} \langle hD \rangle^{-1} u\|_{L^2}) \leq Ch \|u\|_{L_\theta^2}.$$

Combining the two previous estimates, gives an estimate for the second commutator term in (24), i.e.

$$h^2 \|e^{\varphi_\epsilon/h} [\Delta, \chi] (e^{-\varphi_\epsilon/h} \langle hD \rangle^{-1} u)\|_{L^2} \leq Ch \|u\|_{L_\theta^2}.$$

It remains to estimate the first commutator term in (24). I.e. we want to show that

$$\|\chi [P_h, \langle hD \rangle^{-1}] u\|_{L^2} \leq C \|u\|_{L_\theta^2}. \quad (25)$$

Parts of P_h commute with $\langle hD \rangle^{-1}$, so that we are left with

$$\begin{aligned} [P_h, \langle hD \rangle^{-1}] &= [-|d\varphi_\epsilon|^2, \langle hD \rangle^{-1}] + 2h[\nabla \varphi_\epsilon \cdot \nabla, \langle hD \rangle^{-1}] \\ &\quad - h[\Delta \varphi_\epsilon, \langle hD \rangle^{-1}]. \end{aligned} \quad (26)$$

As in the proof of Proposition 3.2, we utilize the asymptotics given by (11) according to which $|d\varphi_\epsilon|^2 = c + O(r^{\theta+\delta/2})$, where c is a constant and $\Delta \varphi_\epsilon = O(r^{-1+\theta+\delta/2})$. This enables us to apply Lemma 3.2 to the first and third commutator in (26), by which we get an improvement in decay, which is crucial. We get that

$$\begin{aligned} \| [|d\varphi_\epsilon|^2, \langle hD \rangle^{-1}] u \|_{L^2} + \| h[\Delta \varphi_\epsilon, \langle hD \rangle^{-1}] \|_{L^2} &\leq Ch \|\langle r \rangle^{\theta+\delta/2-1} u\|_{L^2} \\ &\leq Ch \|u\|_{L_\theta^2}, \end{aligned}$$

where C independent of ϵ . The second commutator term in (26) is

$$\begin{aligned} 2h[\nabla \varphi_\epsilon \cdot \nabla, \langle hD \rangle^{-1}] &= 2\nabla \varphi_\epsilon \cdot [h\nabla, \langle hD \rangle^{-1}] + 2[\nabla \varphi_\epsilon \cdot, \langle hD \rangle^{-1}] h\nabla \\ &= [\nabla \varphi_\epsilon \cdot, \langle hD \rangle^{-1}] h\nabla. \end{aligned}$$

The asymptotics in (11) give that $\nabla \varphi_\epsilon = (\gamma_1, \gamma_2) + (b_1, b_2)$, where γ_j are constants and $b_j = O(r^{\theta+\delta/2})$. The above commutator can be estimated by,

applying Lemma 3.2 to the components of $\nabla\varphi_\epsilon$, giving

$$\begin{aligned} \| [b_j, \langle hD \rangle^{-1}] h \nabla u \|_{L^2} &\leq Ch \langle r \rangle^{\theta+\delta/2-1} h \nabla u \|_{H_{scl}^{-1}} \\ &\leq Ch (\| h \nabla (\langle r \rangle^{\theta+\delta/2-1} u) \|_{H_{scl}^{-1}} + \| [\langle r \rangle^{\theta+\delta/2-1}, h \nabla] u \|_{H_{scl}^{-1}}) \\ &\leq Ch (\| u \|_{L_\theta^2} + h \| \langle r \rangle^{\theta+\delta/2-2} u \|_{H_{scl}^{-1}}) \\ &\leq Ch \| u \|_{L_\theta^2}, \end{aligned}$$

for small h and where C does not depend on ϵ . We thus see that (25) holds. \square

To complete the proof of the previous Proposition we need to prove the following Lemma.

Lemma 3.2. Let $\kappa \geq 0$ and $b(y) \in C^\infty(\mathbb{R}^2)$ satisfies the estimate

$$|\partial_y^\beta b| \leq C_\beta \langle y \rangle^{-\kappa-|\beta|},$$

Then we have the estimate

$$\| [\langle hD \rangle^{-1}, b] u \|_{H_{scl}^{s+2}} \leq Ch \| \langle r \rangle^{-\kappa-1} u \|_{H_{scl}^s},$$

for $u \in C_0^\infty(\mathbb{R}^2)$ and $s \in \mathbb{R}$.

Proof. Let $a(\xi) := \sigma(\langle hD \rangle^{-1}) = \langle \xi \rangle^{-1}$. The composition $\text{Op}_h(a) \text{Op}_h(b)$ can be written as follows

$$\text{Op}_h(c)u = \text{Op}_h(a) \text{Op}_h(b)u = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{iy \cdot \xi} c(y, h\xi; h) \hat{u}(\xi) d\xi,$$

where the symbol c can in turn be written in terms of oscillatory integrals, as

$$c(y, \xi; h) = (2\pi h)^{-2} \int_{\mathbb{R}^2} e^{iz \cdot \xi/h} K(z; h) b(y+z) dz,$$

where

$$K(z; h) := \int_{\mathbb{R}^2} e^{-iz \cdot \eta/h} a(\eta) d\eta.$$

We can split c by the Taylor Theorem as follows

$$c(y, \xi; h) = (2\pi h)^{-2} \int_{\mathbb{R}^2} e^{iz \cdot \xi/h} K(z; h) (b(y) + R_1(y, z)) dz,$$

where R_1 is a remainder term given by

$$R_1(y, z) = \sum_{j=1}^2 z_j \int_0^1 (\partial_j b)(y + \theta z) d\theta.$$

Here $\partial_j b(y)$ denotes the partial derivative of b with respect to the j -th variable.

One sees easily using the fact that¹ $\hat{1}(\eta) = \delta(\eta)$, that

$$c(y, \xi; h) = ab + (2\pi h)^{-2} \int_{\mathbb{R}^2} e^{iz \cdot \xi/h} K(z; h) R_1(y, z) dz.$$

¹here δ is the Dirac delta function

A direct consequence of this is that

$$\sigma([\text{Op}_h(a), \text{Op}_h(b)]) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{iz \cdot \xi} K(hz; h) R_1(y, hz) dz. \quad (27)$$

Define $\mu := h^{-1} \langle y \rangle^{1+\kappa} \sigma([\text{Op}_h(a), \text{Op}_h(b)])$. We will now show that $\mu \in \mathcal{S}^{-2}$. For this we need to check that condition (12) holds, with $m = -2$, i.e. we need to show that

$$|\partial_y^\beta \partial_\xi^\gamma \mu(y, \xi; h)| \leq C_{\beta\gamma} \langle \xi \rangle^{-2}, \quad (28)$$

where $C_{\beta\gamma}$ is independent of h . By the assumption on b and the form of K it suffices show this for the case $\beta = \gamma = 0$.

Splitting the integral in (27) by the triangle inequality into two components with the index $j = 1, 2$ and by integrating by parts, we see that we can estimate

$$I_j := h \left| \int_{\mathbb{R}^2} e^{iz \cdot \xi} \int_{\mathbb{R}^2} e^{-iz \cdot \eta} \partial_j a(\eta) d\eta \int_0^1 (\partial_j b)(y + \theta hz) d\theta dz \right|,$$

to get an estimate for $\sigma([\text{Op}_h(a), \text{Op}_h(b)])$.

We use the abbreviation $B_j(y, z) := \int_0^1 (\partial_j b)(y + \theta z) d\theta$. Then by integrating by parts, we have that

$$\begin{aligned} I_j &= h \left| \int e^{iz \cdot \xi} \int \langle z \rangle^{-2N} e^{-iz \cdot \eta} \langle D_\eta \rangle^{2N} \partial_j a(\eta) d\eta B_j(y, hz) dz \right| \\ &= h \left| \int \int \frac{e^{iz \cdot (\xi - \eta)}}{\langle \xi - \eta \rangle^2} \langle D_\eta \rangle^{2N} \partial_j a(\eta) d\eta \langle D_z \rangle^2 \frac{B_j(y, hz)}{\langle z \rangle^{2N}} dz \right|. \end{aligned}$$

By the Peetre inequality

$$\begin{aligned} I_j &\leq h \left\| \frac{\langle D_\eta \rangle^{2N} \partial_j a}{\langle \xi - \eta \rangle^2} \right\|_{L^\infty(\mathbb{R}_\eta^2)} \left\| \int e^{-iz \cdot \eta} \langle D_z \rangle^2 \frac{B_j(y, hz)}{\langle z \rangle^{2N}} dz \right\|_{L^1(\mathbb{R}_z^2)} \\ &\leq h \langle \xi \rangle^{-2} \left\| \langle \eta \rangle^2 \langle D_\eta \rangle^{2N} \partial_j a \right\|_{L^\infty(\mathbb{R}_\eta^2)} \left\| \int e^{-iz \cdot \eta} \langle D_z \rangle^2 \frac{B_j(y, hz)}{\langle z \rangle^{2N}} dz \right\|_{L^1(\mathbb{R}_z^2)}. \end{aligned}$$

Moreover by the Cauchy-Schwarz inequality we have that

$$\begin{aligned} \left\| \int e^{-iz \cdot \eta} \langle D_z \rangle^2 \frac{B_j(y, hz)}{\langle z \rangle^{2N}} dz \right\|_{L^1} &= \left\| \int \frac{\langle D_z \rangle^{2k}}{\langle \eta \rangle^{2k}} e^{-iz \cdot \eta} \langle D_z \rangle^2 \frac{B_j(y, hz)}{\langle z \rangle^{2N}} dz \right\|_{L^1} \\ &\leq \|\langle \eta \rangle^{-2k}\|_{L^2} \left\| \int e^{-iz \cdot \eta} \langle D_z \rangle^{2k+2} \frac{B_j(y, hz)}{\langle z \rangle^{2N}} dz \right\|_{L^2}. \end{aligned}$$

Thus

$$I_j \leq \frac{C_{k,N} h}{\langle \xi \rangle^2} \|\langle \eta \rangle^2 \langle D_\eta \rangle^{2N} \partial_j a\|_{L^p(\mathbb{R}_\eta^2)} \left\| \frac{1}{\langle \eta \rangle^{2k}} \right\|_{L^2} \left\| \int_0^1 \langle D_z \rangle^{2k+2} \frac{(\partial_j b)(y + \theta hz)}{\langle z \rangle^{2N}} d\theta \right\|_{L^2(\mathbb{R}_z^2)}$$

The above norms become finite when N and k are large enough. By the assumption on b it suffices to show that

$$\left\| \int_0^1 \frac{(\partial_j b)(y + \theta hz)}{\langle z \rangle^{2N}} d\theta \right\|_{L^2(\mathbb{R}_z^2)} \leq C \langle y \rangle^{-\kappa-1}.$$

Indeed, apply the assumption on b and the Peetre inequality we have that $|(\partial_j b)(y + \theta hz)| \leq C \langle y + \theta hz \rangle^{-\kappa-1} \leq C \langle y \rangle^{-\kappa-1} \langle hz \rangle^{|\kappa+1|}$ so the inequality

holds provided that N is large enough. It follows that

$$|\sigma([\text{Op}_h(a), \text{Op}_h(b)])| \leq Ch\langle y \rangle^{-\kappa-1}\langle \xi \rangle^{-2}.$$

This proves (28) and $\mu = h^{-1}\langle y \rangle^{1+\kappa}\sigma([\text{Op}_h(a), \text{Op}_h(b)]) \in \mathcal{S}^{-2}$ which, by Proposition 3.3, means that $\text{Op}_h(\mu) : H_{\delta, scl}^s \rightarrow H_{\delta, scl}^{s+2}$ continuously. Hence

$$\|[\text{Op}_h(a), \text{Op}_h(b)]u\|_{H_{scl}^s} \leq Ch\|\langle r \rangle^{-1-\kappa}u\|_{H_{scl}^{s-2}},$$

which is what we needed to prove. \square

We can combine Proposition 3.4 and 3.1 to obtain a global estimate. To handle the perturbed operator $L_{X,V}$, we need to assume that potentials have decay at least as fast as the weights on the L^2 -norms.

Lemma 3.3. Let φ_ϵ be given by (10). Then for all $V \in x^{1-\frac{\delta}{2}}L^\infty(M_0)$ and $X \in x^{1-\frac{\delta}{2}}W^{1,\infty}(M_0, T^*M_0)$ there exists an $h_0 > 0$, ϵ_0 and $C > 0$ such that for all $0 < h < h_0$, $h \ll \epsilon < \epsilon_0$ and $u \in e^{-\gamma/x}C^\infty(M_0)$, we have

$$\frac{C}{\epsilon}\|x^{1-\frac{\delta}{2}}u\|_{L^2} \leq \sqrt{h}\|e^{\varphi_\epsilon/h}(L_{X,V} - \lambda^2)e^{-\varphi_\epsilon/h}u\|_{H_{scl}^{-1}}.$$

Proof. We first consider only the case when $X = 0$ and $V = 0$. Let $u \in e^{-\gamma/x}C^\infty(M_0)$ and pick $\chi \in C_0^\infty(M_0)$ such that $\chi = 1$ on the compact set containing all the critical points of φ and $\text{supp}(1 - \chi)$ is contained in the ends. By Propositions 3.1 we have the following estimate for χu

$$\begin{aligned} \frac{Ch}{\epsilon}(\sqrt{h}\|\chi u\|_{L^2} + \|d\varphi_\epsilon \chi u\|_{L^2} + \|hd(\chi u)\|_{H_{scl}^{-1}}) &\leq \|e^{\varphi_\epsilon/h}h^2(\Delta - \lambda^2)e^{-\varphi_\epsilon/h}\chi u\|_{H_{scl}^{-1}} \\ &\leq \|\chi e^{\varphi_\epsilon/h}h^2(\Delta - \lambda^2)e^{-\varphi_\epsilon/h}u\|_{H_{scl}^{-1}} + \|e^{\varphi_\epsilon/h}[h^2\Delta, \chi]e^{-\varphi_\epsilon/h}u\|_{H_{scl}^{-1}}. \end{aligned}$$

A limiting argument shows that Proposition 3.4 can be applied to smooth function with exponential decay. Therefore, we have the following estimates for $(1 - \chi)u$

$$\begin{aligned} \frac{Ch}{\epsilon}\|x^{1-\delta/2}(1 - \chi)u\|_{L^2} &\leq \|e^{\varphi_\epsilon/h}h^2(\Delta - \lambda^2)e^{-\varphi_\epsilon/h}(1 - \chi)u\|_{H_{scl}^{-1}} \\ &\leq \|(1 - \chi)e^{\varphi_\epsilon/h}h^2(\Delta - \lambda^2)e^{-\varphi_\epsilon/h}u\|_{H_{scl}^{-1}} + \|e^{\varphi_\epsilon/h}[h^2\Delta, \chi]e^{-\varphi_\epsilon/h}u\|_{H_{scl}^{-1}}. \end{aligned}$$

Adding these two inequalities together we obtain

$$\begin{aligned} \frac{Ch}{\epsilon}(\sqrt{h}\|x^{1-\delta/2}u\|_{L^2} + \|d\varphi_\epsilon \chi u\|_{L^2} + \|x^{1-\delta/2}(1 - \chi)u\|_{L^2} + \|hd(\chi u)\|_{H_{scl}^{-1}}) &\leq \quad (29) \\ &\quad \|e^{\varphi_\epsilon/h}h^2(\Delta - \lambda^2)e^{-\varphi_\epsilon/h}u\|_{H_{scl}^{-1}} + \|e^{\varphi_\epsilon/h}[h^2\Delta, \chi]e^{-\varphi_\epsilon/h}u\|_{H_{scl}^{-1}}. \end{aligned}$$

The next step is to absorb the commutator term on the right-side. To this end we first observe that on the left-side

$$\|x^{1-\delta/2}ud\varphi_\epsilon\|_{L^2} \leq \|\chi ud\varphi_\epsilon\|_{L^2} + \|x^{1-\delta/2}(1 - \chi)u\|_{L^2}$$

while on the right-side

$$\|e^{\varphi_\epsilon/h}[h^2\Delta, \chi]e^{-\varphi_\epsilon/h}u\|_{H_{scl}^{-1}} \leq h\|\tilde{\chi}u\|_{L^2}$$

for some smooth cut-off $\tilde{\chi} \in C_0^\infty(M_0)$ which is equal to 1 on $\text{supp}(d\chi)$ but supported away from the critical points of φ . These two inequalities allows

one to absorb the commutator term on the right-side of (29), when taking $\epsilon > 0$ small enough, to obtain

$$\frac{Ch}{\epsilon}(\sqrt{h}\|x^{1-\delta/2}u\|_{L^2} + \|x^{1-\delta/2}ud\varphi_\epsilon\|_{L^2} + \|hd(\chi u)\|_{H_{scl}^{-1}}) \leq \|e^{\varphi_\epsilon/h}h^2(\Delta - \lambda^2)e^{-\varphi_\epsilon/h}u\|_{H_{scl}^{-1}}.$$

We now replace the Laplacian by the more general operator $L_{X,V}$. Observe that $L_{X,V} - \Delta = 2\langle X, d\cdot \rangle + Q$ for some $Q \in e^{-\gamma/x}L^\infty$ and the zeroth order term can be absorbed to the left-side. Therefore

$$\begin{aligned} & \frac{Ch}{\epsilon}(\sqrt{h}\|x^{1-\delta/2}u\|_{L^2} + \|x^{1-\delta/2}ud\varphi_\epsilon\|_{L^2} + \|hd(\chi u)\|_{H_{scl}^{-1}}) \\ & \leq \|e^{\varphi_\epsilon/h}h^2(L_{X,V} - \lambda^2)e^{-\varphi_\epsilon/h}u\|_{H_{scl}^{-1}} + h\|e^{\varphi_\epsilon/h}\langle X, hd(e^{-\varphi_\epsilon/h}u) \rangle\|_{H_{scl}^{-1}}. \end{aligned}$$

Again we need to absorb the last term on the right-side. This is done by first observing that

$$\begin{aligned} h\|e^{\varphi_\epsilon/h}\langle X, hd(e^{-\varphi_\epsilon/h}u) \rangle\|_{H_{scl}^{-1}} & \leq h\|X\|d\varphi_\epsilon\|u\|_{L^2} + h\|\langle X, hd(\chi u) \rangle\|_{H_{scl}^{-1}} \\ & \quad + h\|\langle X, hd(1 - \chi)u \rangle\|_{H_{scl}^{-1}} \\ & \leq h\|x^{1-\delta/h}ud\varphi_\epsilon\| + h\|hd(\chi u)\|_{H_{scl}^{-1}} \\ & \quad + h\|x^{1-\delta/2}(1 - \chi)u\|_{L^2}. \end{aligned}$$

One sees then that the extra term can indeed be absorbed into the left-side by taking $\epsilon > 0$ small enough. \square

We can utilize the above estimates to obtain an existence result, which is needed when constructing the CGO solutions.

Lemma 3.4. Let $\delta \in (0, 1)$, $V \in x^{1-\frac{\delta}{2}}L^\infty(M_0)$, $X \in x^{1-\frac{\delta}{2}}W^{1,\infty}(M_0, T^*M_0)$ and φ_ϵ as in (10). For all $f \in L^2(M_0)$ and all $h > 0$ small enough, there exists a solution $u \in L^2(M_0)$ to the equation

$$e^{\varphi_\epsilon/h}(L_{X,V} - \lambda^2)e^{-\varphi_\epsilon/h}u = x^{1-\frac{\delta}{2}}f \quad (30)$$

satisfying

$$\|u\|_{L^2} + h\|du\|_{L^2} \leq C\sqrt{h}\|f\|_{L^2}.$$

Proof. Let $L := e^{\varphi_\epsilon/h}(L_{X,V} - \lambda^2)e^{-\varphi_\epsilon/h}$ and consider the linear space

$$\mathcal{H} := \{L^*v \mid v \in C_0^\infty(M_0)\}.$$

Define a linear operator $T : \mathcal{H} \rightarrow \mathbb{C}$ by $T(L^*v) := (x^{1-\delta/2}v, f)_{L^2}$. Lemma 3.3 applies also to L^* , which shows that T is well defined. Observe that $\text{Dom}(T)$ is a linear subspace of $H_{scl}^{-1}(M_0)$. By Lemma 3.3 again, one has that

$$|T(L^*v)| = |(x^{1-\delta/2}v, f)_{L^2}| \leq \|x^{1-\delta/2}v\|_{L^2}\|f\|_{L^2} \leq C\epsilon\sqrt{h}\|L^*v\|_{H_{scl}^{-1}}\|f\|_{L^2}. \quad (31)$$

The map T is hence bounded on the subspace \mathcal{H} in the $H_{scl}^{-1}(M_0)$ norm. By the Hahn-Banach Theorem this map extends to a bounded linear functional on H_{scl}^{-1} with the same norm, which we still denote by T . By duality there exists a $u \in H_{scl}^1(M_0)$, such that $T(w) = \langle u, w \rangle$ for all $w \in H_{scl}^{-1}(M_0)$ where $\langle \cdot, \cdot \rangle$ denotes the duality between H_{scl}^1 and H_{scl}^{-1} . Furthermore u satisfies the

estimate $\|u\|_{H_{scl}^1} = \|T\|_{(H_{scl}^{-1})^*} \leq C\epsilon\sqrt{h}\|f\|_{L^2}$. We then have that for all $v \in C_0^\infty(M_0)$,

$$(x^{1-\delta/2}v, f) = T(L^*v) = (u, L^*v)$$

and this is precisely the statement that u is a weak solution of $Lu = x^{1-\delta/2}f$. \square

Later we conjugate $L_{X,V}$ with an additional function F_A , of the form specified at the end of Section 5. The functions F_A are in particular smooth non-vanishing functions on M_0 , which has the expression $F_A = e^{i\alpha}(1+t)$ with α given by Proposition 2.2, t bounded uniformly away from -1 and in the space $e^{-\gamma/x}W^{1,\infty}(M_0)$ for all $\gamma > 0$. The following Proposition gives a solvability result in terms the additional conjugation.

Proposition 3.5. Let $V \in x^{1-\frac{\delta}{2}}L^\infty(M_0)$, $X \in x^{1-\frac{\delta}{2}}W^{1,\infty}(M_0, T^*M_0)$ and let $f \in x^J L^2$ for some $J \in \mathbb{R}$. There exists solutions $w \in H_{loc}^1$ to the equation

$$e^{-\Phi/h}F_A^{-1}(L_{X,V} - \lambda^2)e^{\Phi/h}F_A w = f, \quad (32)$$

where Φ is as in (9), which satisfies, the estimate

$$\|e^{\varphi_0/\epsilon}w\|_{L^2} + h\|e^{\varphi_0/\epsilon}dw\|_{L^2} \leq C\sqrt{h}\|x^{-J}f\|_{L^2},$$

where φ_0 is as required in definition (10).

Proof. By the assumption on the form of F_A it suffices to show this for $F_A = e^{i\alpha}$. Let $f \in x^J L^2$. Since $F_A e^{-\text{Re}(i\alpha)}$ is bounded and that $e^{\varphi_0/\epsilon}$ decays faster than any polynomial in x , we have that

$$e^{(\varphi-\varphi_\epsilon)/h}F_A f = F_A e^{-\text{Re}(i\alpha)}e^{\varphi_0/\epsilon}f \in x^{1-\frac{\delta}{2}}L^2(M_0).$$

By Lemma 3.4 there is a solution u to the equation

$$e^{-\varphi_\epsilon/h}(L_{X,V} - \lambda^2)e^{\varphi_\epsilon/h}u = e^{(\varphi-\varphi_\epsilon)/h}F_A e^{i\psi/h}f.$$

Define $v := e^{(\varphi_\epsilon-\varphi)/h}F_A^{-1}u = e^{\text{Re}(i\alpha)}F_A^{-1}e^{-\varphi_0/\epsilon}u$. It follows that v solves

$$e^{-\varphi/h}F_A^{-1}(L_{X,V} - \lambda^2)e^{\varphi/h}F_A v = e^{i\psi/h}f.$$

The norm estimate of Lemma 3.4 gives furthermore that

$$\|u\|_{L^2} + h\|du\|_{L^2} \leq C\sqrt{h}\|e^{(\varphi-\varphi_\epsilon)/h}F_A f\|_{L^2} \leq C\sqrt{h}\|x^{-J}f\|_{L^2}, \quad (33)$$

where the second inequality is obtained from the fact that multiplication by $e^{(\varphi-\varphi_\epsilon)/h}F_A x^J = F_A e^{-\text{Re}(i\alpha)}e^{\varphi_0/\epsilon}x^J \in L^\infty(M_0)$ is L^2 -continuous on M_0 . Next we estimate the right hand side of (33) from below. Firstly

$$\|u\|_{L^2} = \|F_A e^{-\text{Re}(i\alpha)}e^{\varphi_0/\epsilon}v\|_{L^2} \geq C\|e^{\varphi_0/\epsilon}v\|_{L^2}, \quad (34)$$

since $e^{\text{Re}(i\alpha)}/F_A \in L^\infty(M_0)$. Expanding the derivative and use the assumption that $F_A = e^{i\alpha}$ gives that

$$du = F_A e^{-\text{Re}(i\alpha)}e^{\varphi_0/\epsilon}((d\varphi_0/\epsilon - i\text{Re}(d\alpha))v + dv).$$

Since $F_A^{-1}e^{\text{Re}(i\alpha)} \in L^\infty(M_0)$, we have that

$$\begin{aligned} \|du\|_{L^2} &\geq C(\|e^{\varphi_0/\epsilon}dv\|_{L^2} - \|e^{\varphi_0/\epsilon}(d\varphi_0/\epsilon - i\text{Re}(d\alpha))v\|_{L^2}) \\ &\geq C(\|e^{\varphi_0/\epsilon}dv\|_{L^2} - \|e^{\varphi_0/\epsilon}v\|_{L^2}), \end{aligned}$$

where in the second step, we used that $d\varphi_0/\epsilon - i\operatorname{Re}(d\alpha) \in L^\infty(M_0)$, which holds because of (11) and because of the expression of α given in Proposition 2.2. This together with (34), gives

$$h^{-1}\|e^{\varphi_0/\epsilon}v\|_{L^2} + \|e^{\varphi_0/\epsilon}dv\|_{L^2} \leq C(h^{-1}\|u\|_{L^2} + \|du\|_{L^2}), \quad (35)$$

when h is small. From (33) we get that

$$\|e^{\varphi_0/\epsilon}v\|_{L^2} + h\|e^{\varphi_0/\epsilon}dv\|_{L^2} \leq C\sqrt{h}\|x^{-J}f\|_{L^2}.$$

Finally setting $w := e^{-i\psi/h}v$, we see that w solves

$$e^{-\Phi/h}F_A^{-1}(L_{X,V} - \lambda^2)e^{\Phi/h}F_A w = f,$$

and that we have the estimate of the claim. \square

4. SCATTERING BY $L_{X,V}$ ON SURFACES WITH EUCLIDEAN ENDS

In this section we construct the scattering matrix through the use of the Poisson operator for the operator $L_{X,V}$ on surfaces with Euclidean ends. Furthermore we will show that the range of the Poisson operator is dense in some suitably defined exponentially weighted solution spaces:

Proposition 4.1. There exists an operator $P_{X,V}(\lambda) : C^\infty(\partial M_0) \rightarrow x^{-\tau}H^1(M_0)$ satisfying for all $f_+ \in C^\infty(\partial M_0)$ there exists a unique $f_- \in C^\infty(\partial M_0)$ such that

$$P_{X,V}(\lambda)f_+ - (x^{1/2}e^{\lambda/x}f_+ + x^{1/2}e^{-\lambda/x}f_-) \in L^2(M_0), \quad (L_{X,V} - \lambda^2)P_{X,V}(\lambda)f_+ = 0. \quad (36)$$

We define the scattering matrix $S_{X,V}(\lambda)$ by $S_{X,V}(\lambda)f_+ := f_-$.

Proposition 4.2. Let $0 < \gamma < \gamma' < \gamma_0$. If $X \in e^{-\gamma_0/x}L^\infty$ and $V \in e^{-\gamma_0/x}L^\infty$ the set

$$\{P_{X,V}(\lambda)f \mid f \in C^\infty(\partial M_0)\}$$

is dense in the null space of $L_{X,V} - \lambda^2$ in $e^{\gamma/x}L^2$ with respect to the $e^{\gamma'/x}L^2$ topology.

We first define the free resolvent $R_0(\lambda) : L^2 \rightarrow H^2$ on \mathbb{R}^2 for λ on the lower half of the complex plane. If $A > 0$ then for all $\gamma > A$ this resolvent extends as a holomorphic family of operators $R_0(\lambda) : e^{-\gamma/x}L^2 \rightarrow e^{\gamma/x}H^2$ as λ vary over the set $\{\lambda \mid \operatorname{Im}(\lambda) < A, \lambda \notin i\mathbb{R}^+ \cup 0\}$. Direct computation also yields that for all $\tau > 1/2$ one has $R_0(\lambda) : x^\tau L^2 \rightarrow x^{-\tau}H^2$ when λ lies on the positive real axis. This fact is usually stated in weighted L^2 spaces but the Sobolev estimate can be obtained by writing

$$(d^*d + 1)R_0(\lambda) = Id + (\lambda^2 + 1)R_0(\lambda).$$

We generalize this statement for the operator $L_{X,V}$ on the surface M_0 :

Lemma 4.1. If $A > 0$ then for all $\gamma > A$ the resolvent $R_{X,V}(\lambda) := (L_{X,V} - \lambda^2)^{-1}$ is defined as a meromorphic family of operators mapping $e^{-\gamma/x}L^2 \rightarrow e^{\gamma/x}L^2$ over the set $\{\lambda \mid \operatorname{Im}(\lambda) < A, \lambda \notin i\mathbb{R}^+ \cup 0\}$. Furthermore, if $\lambda \in \mathbb{R}^+$ is not a pole of $R_{X,V}(\lambda)$ then it is a bounded map from $x^\tau L^2 \rightarrow x^{-\tau}H^1$ for any $\tau > 1/2$.

Proof. We let $\chi \in C_0^\infty(M_0)$ be a smooth function such that $1 - \chi$ is supported near E_j . We let $\chi_0, \chi_1 \in C_0^\infty(M_0)$ be smooth functions such that $\chi_0 = 1$ on the support of χ and $1 - \chi_1 = 1$ on the support of $1 - \chi$. We observe that if we chose λ_0 to have a large negative imaginary part, then for the parametrix

$$E(\lambda) := (1 - \chi_1)R_0(\lambda)(1 - \chi) + \chi_0 R_0(\lambda_0)\chi$$

we have $(L_{X,V} - \lambda^2)E(\lambda) = I + K(\lambda)$ where $K(\lambda) : x^\tau L^2(M_0) \rightarrow x^\tau H^1(M_0)$ is given by

$$K(\lambda) = ([\Delta_{g_0}, \chi_1] - (\lambda^2 - \lambda_0^2)\chi_1)R_0(\lambda_0)\chi - [\Delta_{g_0}, \chi_0]R_0(\lambda)(1 - \chi) + (X^\sharp + \star dX + |X|^2 + V)E(\lambda)$$

and X^\sharp denotes differentiation with respect to the vector field obtained by raising the index on the 1-form X . By the mapping properties of

$$R_0(\lambda) : x^\tau L^2(\mathbb{R}^2) \rightarrow x^{-\tau} H^2(\mathbb{R}^2), \quad R_0(\lambda_0) : L^2(M_0) \rightarrow H^2(M_0),$$

and the super-exponential rates of decay of X and V , we have that $K(\lambda)$ is a holomorphic family of compact operators from $e^{-\gamma/x} L^2$ to itself. If $\lambda = \lambda_0$ has a large negative imaginary part, then $I + K(\lambda)$ is invertible by Neumann series. Therefore, by the analytic Fredholm theorem $(I + K(\lambda))^{-1}$ is a meromorphic family of operators from $e^{-\gamma/x} L^2$ to itself as λ varies over the region $\{\lambda \mid \text{Im}(\lambda) < A, \lambda \notin i\mathbb{R}^+ \cup 0\}$. Setting $R_{X,V}(\lambda) := E(\lambda)(1 + K(\lambda))^{-1}$ proves the portion of the Lemma for the exponentially weighted L^2 spaces.

For the resolvent acting on $x^\tau L^2$, we need to show that $1 + K(\lambda)$ is invertible on $x^\tau L^2$ for $\tau > 1/2$. Similar argument as before shows that $K(\lambda)$ is compact on $x^\tau L^2$ and therefore the invertibility of $1 + K(\lambda)$ at a given $\lambda \in \mathbb{R}^+$ can be deduced from the triviality of its null-space. Indeed, if λ is not a pole of the resolvent $R_{X,V}(\lambda)$ acting on $e^{-\gamma/x} L^2$, then $1 + K(\lambda)$ is invertible on $e^{-\gamma/x} L^2$. Suppose $u \in x^\tau L^2$ is in the null-space of $1 + K(\lambda)$ then it is actually an element of $e^{-\gamma/x} L^2$ by the decay properties of the coefficients in $K(\lambda)$. As $1 + K(\lambda)$ is invertible on $e^{-\gamma/x} L^2$, we have that $u = 0$. Therefore, $R_{X,V}(\lambda) = E(\lambda)(1 + K(\lambda))^{-1}$ is a resolvent mapping $x^\tau L^2 \rightarrow x^{-\tau} H^1$ when $\lambda \in \mathbb{R}^+$ is not a pole. \square

It is well-known ([20]) that for all $f \in e^{-\gamma/x} L^2(\mathbb{R}^2)$ the free resolvent has asymptotic given by

$$R_0(\lambda)f - x^{1/2}e^{-i\lambda/x}v \in L^2(\mathbb{R}^2)$$

for some smooth function $v \in C^\infty(S^1)$. By the construction of $E(\lambda)$ and $R_{X,V}(\lambda)$ this gives the expansion

$$E(\lambda)f - x^{1/2}e^{-i\lambda/x}v \in L^2(M_0), \quad R_{X,V}(\lambda)f - x^{1/2}e^{-i\lambda/x}v' \in L^2(M_0) \quad (37)$$

for some $v, v' \in C^\infty(\partial M_0)$.

We would like to prove that the resolvent has no poles on \mathbb{R}^+ . Following the exposition of [20] we first prove that

Lemma 4.2. The poles of resolvent $R_{X,V}(\lambda)$, are precisely the values λ for which there exists a nontrivial solution $u \in x^{-\tau} H^1(M_0)$ of the equation $(L_{X,V} - \lambda^2)u = 0$ satisfying $u - x^{1/2}e^{-i\lambda/x}v \in L^2(M_0)$ for some smooth function $v \in C^\infty(\partial M_0)$.

Proof. If λ' is a pole of $R_{X,V}(\lambda)$ then it must be a pole of $(1 + K(\lambda))^{-1}$ as the parametrix $E(\lambda)$ is holomorphic. Therefore, there exists $f \in e^{-\gamma/x} L^2$ for which $(1 + K(\lambda))^{-1} f$ has a pole at λ' with residue $u' \in e^{-\gamma/x} L^2$. Using the fact that $(1 + K(\lambda))(1 + K(\lambda))^{-1} f = f$ we have that $u' = -K(\lambda')u'$. Therefore, if we set $u := E(\lambda')u'$ then $(L_{X,V} - \lambda'^2)u = (1 + K(\lambda'))u' = 0$ and the asymptotic of u can be derived from (37). \square

We now show that the embedded eigenvalue obtained in Lemma 4.2 must be trivial. To this end we first derive the boundary pairing identity

Lemma 4.3. For $\lambda > 0$ and $X, V \in e^{-\gamma/x} L^\infty(M_0)$, if $u_\pm \in x^{-\tau} H^1(M_0)$ for some $\tau > \frac{1}{2}$ and $(L_{X,V} - \lambda^2)u_\pm \in x^\tau L^2(M_0)$ with

$$u_\pm - x^{1/2} e^{i\lambda/x} f_{\pm+} - x^{1/2} e^{-i\lambda/x} f_{\pm-} \in L^2(M_0)$$

then we have the integral identity

$$\langle (L_{X,V} - \lambda^2)u_+, u_- \rangle - \langle u_+, (L_{X,V} - \lambda^2)u_- \rangle = 2i\lambda \int_{\partial M_0} (f_{++}\bar{f}_{-+} - f_{+-}\bar{f}_{--})$$

where the volume form on ∂M_0 is induced by the metric $x^2 g_0|_{T\partial M}$.

Proof. It suffices to prove this for Δ_{g_0} in place of $L_{X,V}$ and use the fact that $L_{X,V} - \Delta_{g_0}$ is a symmetric first order differential operator with super-exponential decaying coefficients.

If $u_\pm \in x^{-\tau} H^1$ with $(\Delta_{g_0} - \lambda^2)u_\pm \in x^\tau L^2$ and

$$r_\pm := u_\pm - x^{1/2} e^{i\lambda/x} f_{\pm+} - x^{1/2} e^{-i\lambda/x} f_{\pm-} \in L^2(M_0)$$

then one can deduce that $r_\pm \in H^2(M_0)$. Therefore, if for $\epsilon > 0$ small we denote $\langle f, g \rangle_{x>\epsilon} := \int_{\{x>\epsilon\}} f \bar{g} d\text{vol}_{g_0}$, we have

$$\begin{aligned} & \langle (\Delta_{g_0} - \lambda^2)u_+, u_- \rangle_{x>\epsilon} - \langle u_+, (\Delta_{g_0} - \lambda^2)u_- \rangle_{x>\epsilon} \\ &= \int_{\{x=\epsilon\}} \bar{u}_- \partial_\nu u_+ - u_+ \partial_\nu \bar{u}_- \\ &= I_\epsilon + \epsilon^{-1/2} \int_{\{x=\epsilon\}} (a_+(r_+ + \partial_\nu r_+) + a_-(r_- + \partial_\nu r_-)) \end{aligned}$$

where $\lim_{\epsilon \rightarrow 0} I_\epsilon = 2i\lambda \int_{\partial M_0} (f_{++}\bar{f}_{-+} - f_{+-}\bar{f}_{--})$ with the volume form induced by the metric $x^2 g_0|_{T\partial M_0}$ and a_\pm are $L^\infty(M_0)$ functions. As $r_\pm \in H^2(M_0)$ we can deduce that there exists a sequence of $\epsilon_j \rightarrow 0$ such that

$$\int_{\{x=\epsilon\}} |(a_+(r_+ + \partial_\nu r_+) + a_-(r_- + \partial_\nu r_-))| \leq \epsilon_j.$$

Taking this sequence $\epsilon_j \rightarrow 0$ and use the fact that $(u_\pm)(\Delta_{g_0} - \lambda^2)u_\mp \in L^1(M_0)$ by assumption allows us to arrive at the desired integral identity. \square

We are now in a position to show that the embedded function $u = x^{1/2} e^{i\lambda/x} v + L^2(M_0)$ constructed in Lemma 4.2 is trivial when $\lambda \in \mathbb{R}^+$ by repeating an argument in [25]. Indeed, by setting $u_+ = u_- = u$ in Lemma 4.3, we see that $v = 0$ and therefore $u \in H^2(M_0)$. Let $\chi \in C_0^\infty(M_0)$ be a smooth compactly supported function such that $1 - \chi$ is only supported in the Euclidean ends and define $u_\chi := (1 - \chi)u$ to be the H^2 function

defined on the disjoint union of finitely many copies of \mathbb{R}^2 . From the super-exponential decay of the coefficients of $L_{X,V}$, we can use Paley-Weiner to conclude that $(|\xi|^2 - \lambda^2)\hat{u}_\chi$ extends to a holomorphic function $g(\xi + i\eta)$ on \mathbb{C}^2 which satisfies the bound

$$\sup_{|\eta| \leq \gamma} \|g(\cdot + i\eta)\|_{L^2} \leq C_\gamma, \quad \forall \gamma > 0.$$

The fact that $\hat{u}_\chi \in L^2$ forces g to vanish on the real variety $\{\xi \in \mathbb{R}^2 \mid \xi \cdot \xi - \lambda^2 = 0\}$ and therefore vanish on the complex codimension one variety $\{\zeta \in \mathbb{C}^2 \mid \zeta \cdot \zeta - \lambda^2 = 0\}$ (see proof of Lemma 2.5 [25]). One sees then that for all multi-indices β with $|\beta| \leq 2$ the function $\xi^\beta \hat{u}_\chi$ extends to a holomorphic function on \mathbb{C}^2 which satisfies the bound

$$\sup_{|\eta| \leq \gamma} \|(\cdot + i\eta)^\beta \hat{u}_\chi(\cdot + i\eta)\|_{L^2} \leq C_\gamma, \quad \forall \gamma > 0.$$

Paley-Weiner then shows that $u \in e^{-\gamma/x} H^2(M_0)$ for all $\gamma > 0$. Applying the Carleman estimate in Proposition 3.3 shows that $u = 0$.

A direct consequence of this discussion in conjunction with Lemma 4.2 yields the following

Corollary 4.1. There does not exist nontrivial solutions to

$$(L_{X,V} - \lambda^2)u_\pm = 0$$

of the form $u_\pm = x^{1/2} e^{\pm i\lambda/x} v_\pm + L^2(M_0)$ for some $v_\pm \in C^\infty(\partial M_0)$. Furthermore, the poles of the resolvent $R_{X,V}(\lambda)$ does not lie on the positive real axis.

Proof of Proposition 4.1 We set

$$P_{X,V}(\lambda) := (1 - \chi)P_0(\lambda) - R_{X,V}(\lambda)(L_{X,V} - \lambda^2)(1 - \chi)P_0(\lambda)$$

where $P_0(\lambda)$ is the free Poisson kernel on \mathbb{R}^2 . The asymptotic expansion of the operator $P_{X,V}(\lambda)$ is then given by (37) and the expansion for $P_0(\lambda)$. The uniqueness of the expansion in (36) comes from Corollary 4.1. \square

We are now in a position to show that the range of the Poisson operator is dense in the solution space of exponentially growing solutions.

Proof of Proposition 4.2 Let $w \in e^{-\gamma'/x} L^2$ be orthogonal to the range of $P_{X,V}(\lambda)$ so that $\langle w, P_{X,V}(\lambda)f_+ \rangle = 0$ for all $f_+ \in C^\infty(\partial M_0)$. We need to show that $\langle u, w \rangle = 0$ for all $u \in e^{\gamma'/x} L^2$ such that $(L_{X,V} - \lambda^2)u = 0$.

To this end consider the function $v := R_{X,V}(\lambda)w = x^{1/2} e^{i\lambda/x} f + L^2(M_0)$ for some $f \in C^\infty(\partial M_0)$. Applying integral identity in Lemma 4.3 with $u_+ = P_{X,V}(\lambda)f_+$ and $u_- = v$ we see that $\int_{\partial M_0} f_+ \bar{f} = 0$ for all $f_+ \in C^\infty(\partial M_0)$. This means that v is an L^2 solution to $(\Delta_{g_0} - \lambda^2)v \in e^{-\gamma'/x} L^2$. If we choose smooth cutoff $\chi \in C_0^\infty(M_0)$ such that $1 - \chi$ is only supported in the Euclidean ends, this would mean that $v_\chi := (1 - \chi)v \in L^2$ solves $(\Delta - \lambda^2)v_\chi \in e^{-\gamma'/x} L^2$ in \mathbb{R}^2 . Repeating the argument made in proving Corollary 4.1 we see that v_χ (and therefore v) is an element of $e^{-\gamma'/x} H^2$.

Now let $u \in e^{\gamma'/x} L^2$ such that $(L_{X,V} - \lambda^2)u = 0$. We may write

$$\langle w, u \rangle = \langle (L_{X,V} - \lambda^2)R_{X,V}(\lambda)w, u \rangle = \langle (L_{X,V} - \lambda^2)v, u \rangle = \langle v, (L_{X,V} - \lambda^2)u \rangle = 0$$

where the integration-by-parts performed in the last step is permitted since $v \in e^{-\gamma'/x} H^2$ for some $\gamma' > \gamma > 0$. \square

5. BOUNDARY IDENTIFIABILITY AT INFINITY

Proposition 5.1. Let $\alpha_j \in x^{-J} H^{4+\epsilon_0}$, $2 > J > 1$, be solutions to $\bar{\partial}\alpha_j = A_j := \pi_{0,1} X_j \in e^{-\gamma/x} H^{3+\epsilon_0}(M_0)$ constructed in Proposition 2.2. Assuming that $S_{X_1, V_1}(\lambda) = S_{X_2, V_2}(\lambda)$, there exists a non-vanishing holomorphic function Ψ satisfying

$$\Psi - e^{i(\alpha_1 - \alpha_2)} \in e^{-\gamma/x} H^{3+\epsilon_0}(M_0) \quad (38)$$

for all $\gamma > 0$.

We will split this into several Lemmas. In all of them we assume without stating that $S_{X_1, V_1}(\lambda) = S_{X_2, V_2}(\lambda)$.

Lemma 5.1. Let $f \in H_b^m(M_0)$ be a function satisfying

$$\bar{\partial}f \in e^{-\gamma/x} H_b^m(M_0; {}^b T_{0,1}^* M_0)$$

for all $\gamma > 0$. Suppose $f \in x^J H_b^m(M_0)$ for all $J \in \mathbb{R}$, then $f \in e^{-\gamma/x} H_b^m(M_0)$ for all $\gamma > 0$.

Proof By localizing f with cutoff functions near the Euclidean ends E_j and arguing each individual ends separately, we may assume without loss of generality that $M_0 = \mathbb{C}$ on which we use the standard variable z . Denote by

$$u := \bar{\partial}f \in e^{-\gamma|z|} H^m(\mathbb{C}), \quad \forall \gamma > 0. \quad (39)$$

It suffices to prove that $f \in e^{-\gamma|z|} L^2(\mathbb{C})$ as the general Sobolev space result follows by considering $f_j := \partial_{x_j} f$ which satisfies $\bar{\partial}f_j = \partial_{x_j} u \in e^{-\gamma|z|} H^{m-1}(\mathbb{C})$.

Taking Fourier Transform of (39) we have that since $f \in |z|^{-J} H^m$ for all $J \in \mathbb{R}$,

$$(\xi_1 + i\xi_2)\hat{f}(\xi) = \hat{u}(\xi), \quad \hat{f} \in C^\infty(\mathbb{R}^2) \quad (40)$$

which gives us a condition at the origin that will be useful later. By Paley-Weiner $\hat{u}(\xi) = \hat{u}(\xi_1, \xi_2)$ extends to be a holomorphic function on \mathbb{C}^2 of two complex variables $\hat{u}(\zeta_1, \zeta_2)$ with $\zeta_j = \xi_j + i\eta_j \in \mathbb{C}$ with $\xi_j = \text{Re}(\zeta_j)$ and $\eta_j = \text{Im}(\zeta_j)$ (sometimes we write $\hat{u}(\zeta_1, \zeta_2) = \hat{u}(\xi + i\eta)$). Furthermore, it satisfies, by (39) and Paley-Wiener,

$$\sup_{|\eta| \leq \gamma} \|\hat{u}(\cdot + i\eta)\|_{L^2}^2 < \infty \quad \forall \gamma \geq 0. \quad (41)$$

We will prove that $\hat{u}(\zeta_1, \zeta_2)$ has power series expansion around the origin of the form

$$\hat{u}(\zeta_1, \zeta_2) = \sum_{j=1, k=0}^{\infty} c_{j,k} (\zeta_1 + i\zeta_2)^j (\zeta_1 - i\zeta_2)^k. \quad (42)$$

Notice that the index j starts at 1 rather than 0. If (42) holds then $\hat{f}(\xi_1, \xi_2)$ would by the removable singularities theorem have a holomorphic extension onto \mathbb{C}^2 given by $\frac{\hat{u}(\zeta_1, \zeta_2)}{\zeta_1 + i\zeta_2}$. (See e.g. Theorem 7.3.3 in [18]).

We proceed to show (42). By the fact that $\hat{u}(\zeta_1, \zeta_2)$ is entire on \mathbb{C}^2 it has a convergent power series expansion in powers of ζ_1 and ζ_2 which we can write as

$$\hat{u}(\zeta_1, \zeta_2) = \sum_{j,k=0}^{\infty} c_{j,k} (\zeta_1 + i\zeta_2)^j (\zeta_1 - i\zeta_2)^k.$$

Setting $\eta_1 = \eta_2 = 0$ so that $\zeta_j = \xi_j$ we have that, by denoting $\xi = \xi_1 + i\xi_2$,

$$\hat{u}(\xi_1, \xi_2) = \sum_{j,k=0}^{\infty} c_{j,k} \xi^j \bar{\xi}^k.$$

We observe that (42) is equivalent to the fact that the above expansion has $c_{0,n} = 0$ for all n . To this end, (40) reads

$$\xi \hat{f}(\xi_1, \xi_2) = \sum_{j,k=0}^{\infty} c_{j,k} \xi^j \bar{\xi}^k$$

for \hat{f} smooth near the origin which immediately gives $c_{0,0} = 0$. Observe that since \hat{f} is smooth, one can also divide by ξ to get the smooth function

$$\hat{f}(\xi_1, \xi_2) = \frac{1}{\xi} \sum_{j,k=0}^{\infty} c_{j,k} \xi^j \bar{\xi}^k.$$

The right hand side is a-priori defined only on the punctured plane but extends smoothly to \mathbb{C} due to the smoothness of \hat{f} . We will now hit both sides with the operator $\xi(\frac{\partial}{\partial \xi})^n$ and take $\xi \rightarrow 0$ to get $0 = c_{0,n}$ for all $n \geq 1$ and (42) is established.

It remains to apply Paley-Wiener to conclude the super-exponentially decay of f . To do so, one needs to check

$$\sup_{|\eta| \leq \gamma} \|\hat{f}(\cdot + i\eta)\|_{L^2}^2 < \infty, \quad \forall \gamma \geq 0.$$

On the strip $|\eta| \leq \gamma$ the vanishing set of $\zeta_1 + i\zeta_2$ is contained in a compact rectangle. On this rectangle $\hat{f}(\zeta_1, \zeta_2)$ is of course bounded. Outside of this rectangle the estimate comes from the fact that $\hat{f} = \frac{\hat{u}}{\zeta_1 + i\zeta_2}$ and the estimate (41). \square

Lemma 5.2. Let f be a smooth function on M_0 satisfying

$$\bar{\partial}f \in e^{-\gamma/x} H_b^m(M_0; {}^bT_{0,1}^* M_0) \quad \forall \gamma > 0.$$

Suppose for all $J \in \mathbb{R}$,

$$\int_{M_0} \langle \bar{\partial}f, \eta \rangle_{g_0} d\text{vol}_{g_0} = 0, \forall \eta \in x^{-J} L^2(M_0; T^* M_0), \quad \bar{\partial}^* \eta = 0 \quad (43)$$

then there exists a holomorphic function Ψ such that $\Psi - f \in e^{-\gamma/x} H_b^m(M_0)$ for all $\gamma > 0$.

Proof We first find a solution U to the equation

$$\bar{\partial}U = -\bar{\partial}f, \quad U \in x^J H_b^m(M_0) \quad \forall J \in \mathbb{R}. \quad (44)$$

Once such a solution is constructed, the proof is complete by evoking Lemma 5.1 to conclude that U belongs to $e^{-\gamma/x} H_b^m(M_0)$ for all $\gamma > 0$. To this end,

as $\bar{\partial}f$ decays super-exponentially, we may consider $\bar{\partial}f$ to be a $x^J H_b^m$ section of ${}^b T^* M_0$ for all $J \in \mathbb{R}$ and observe that by (4) it is an element of $R_{J,m}(\bar{\partial})$ if and only if

$$\int_{M_0} \langle \bar{\partial}f, \eta \rangle_{g_b} d\text{vol}_{g_b} = 0, \forall \eta \in N_{-J,m}({}^b \bar{\partial}^*).$$

Using the relation (5) combined with the fact that

$$\langle \bar{\partial}f, \eta \rangle_{g_0} d\text{vol}_{g_0} = \langle \bar{\partial}f, \eta \rangle_{g_b} d\text{vol}_{g_b}$$

one sees that the condition given by (43) does indeed imply the above orthogonality condition for all $J \in \mathbb{R}$.

Therefore for each $J \in \mathbb{R} \setminus \mathbb{Z}$ one can find a solution $U_J \in x^J H_b^m(M_0)$ solving $\bar{\partial}U_J = -\bar{\partial}f$. Since the difference of two such solutions are holomorphic, uniqueness follows for $J \notin \mathbb{Z}$ large by standard arguments for holomorphic functions. Therefore, as $x^J H_b^m \subset x^{J'} H_b^m$ for $J \geq J'$, there exists a unique solution $U \in \bigcap_{J \in \mathbb{R}} x^J L^2$ of $\bar{\partial}U = -\bar{\partial}f$. This shows that (44) has a unique solution and the proof is complete. \square

Remark 5.1. Note that in neither the statement nor the proof of this Lemma is it required for f to have a polyhomogenous expansion.

By Lemma 5.2 we see that to prove Proposition 5.1 it suffices to show that $e^{i(\alpha_1 - \alpha_2)}$ satisfies the orthogonal condition (43). To this end we first derive the following identity:

Lemma 5.3. Let $u_j \in e^{\gamma/x} H^1(M_0)$ be solutions to $L_{X_j, V_j} u_j = 0$ then the integral identity holds

$$0 = \int_{M_0} \bar{u}_1 (A_1 - A_2) \wedge \partial u_2 + \bar{u}_1 (\bar{A}_1 - \bar{A}_2) \wedge \bar{\partial} u_2 + \bar{u}_1 (Q_1 - Q_2) u_2$$

Proof. By Lemma 4.3 and the fact that $S_{X_1, V_1}(\lambda) = S_{X_2, V_2}(\lambda)$ we have that the above identity holds for $u_j \in x^{-\tau} H^1(M_0)$ in the range of $P_{X_j, V_j}(\lambda)$. We first fix u_2 and use the $e^{\gamma/x} L^2$ density result of Proposition 4.2 to take the limit in u_1 to conclude that the above identity holds for u_2 in the range of $P_{X_2, V_2}(\lambda)$ and solutions $u_1 \in e^{\gamma/x} H^1(M_0)$. Since now u_1 has $e^{\gamma/x} H^1(M_0)$ regularity we can take the limit in u_2 in the $e^{\gamma/x} L^2$ topology to obtain the result. \square

Proof of Proposition 5.1:

We begin by choosing Φ a holomorphic morse function which grows linearly at each end; this function exists by Lemma 2.3. Let $\text{Crit}(\Phi) := \{p_0, \dots, p_n\}$ be the critical points of Φ and, for some $J \in \mathbb{R}$, let $b \in x^J L^2(M_0)$ be an antiholomorphic 1-form on M_0 which vanishes to third order on points in $\text{Crit}(\Phi)$. Consider the ansatz $u_0 = h e^{\bar{\Phi}/h} e^{-i\bar{\alpha}_2} \frac{b}{\bar{\partial}\Phi}$. By writing

$$L_{X,V} = e^{-i\bar{\alpha}} \bar{\partial}^* |e^{-i\alpha}|^2 \bar{\partial} e^{i\alpha} + Q$$

for $Q = |X|^2 + V + dX \in e^{-\gamma/x} L^\infty$ for all $\gamma \in \mathbb{R}$, we see that u_0 solves

$$(L_{X_2, V_2} - \lambda^2) u_0 = h e^{\bar{\Phi}/h} e^{-i\bar{\alpha}_2} f$$

with $f \in x^{J'} L^2(M_0)$ for some $J' \in \mathbb{R}$. We now apply Proposition 3.5 to obtain a solution to $(L_{X_2, V_2} - \lambda^2)u_2 = 0$ of the form

$$u_2 = u_0 + e^{\bar{\Phi}/h} e^{-i\bar{\alpha}_2} r_2$$

with r_2 satisfying the estimate $\|e^{\varphi_0/\epsilon} r_2\| + h\|e^{\varphi_0/\epsilon} dr_2\| \leq h\sqrt{h}C\|x^{-J'} f\|$ where φ_0 is as required in definition (10). For the solution $(L_{X_1, V_1} - \lambda^2)u_1 = 0$ we use the ansatz

$$(L_{X_1, V_1} - \lambda^2)e^{\Phi/h} e^{-i\alpha_1} = e^{\Phi/h} e^{-i\alpha_1} f$$

with $f \in x^J L^2$ for some $J \in \mathbb{R}$. Proposition 3.5 again applies to obtain a solution to $(L_{X_1, V_1} - \lambda^2)u_1 = 0$ of the form

$$u_1 = e^{\Phi/h} e^{-i\alpha_1} + e^{\Phi/h} e^{-i\alpha_1} r_1.$$

with r_1 satisfying the estimate $\|e^{\varphi_0/\epsilon} r_1\| + h\|e^{\varphi_0/\epsilon} dr_1\| \leq \sqrt{h}C\|x^{-J} f\|$ where φ_0 is as required in definition (10).

We now substitute these solutions into the identity in Lemma 5.3 to obtain, after taking $h \rightarrow 0$,

$$0 = \int_{M_0} \langle \bar{\partial} e^{i(\alpha_1 - \alpha_2)}, b \rangle d\text{vol}_{g_0}$$

for all anti-holomorphic 1-forms b vanishing to third order at $\text{Crit}(\Phi) = \{p_0, \dots, p_n\}$ which are in the space $x^J L^2(M_0)$ for some $J \in \mathbb{R}$.

We do not have the orthogonal condition (43) for all anti-holomorphic 1-forms yet because of the restricted vanishing condition. We will get rid of the vanishing condition one point at a time starting with p_0 . To this end, we use Lemma 2.1 and Corollary 2.1 to construct a holomorphic Morse function $\tilde{\Phi}$ for which $p_0 \notin \text{Crit}(\tilde{\Phi}) := \{\tilde{p}_0, \dots, \tilde{p}_m\}$. Repeating the above argument for $\tilde{\Phi}$ we have that

$$0 = \int_{M_0} \langle \bar{\partial} e^{i(\alpha_1 - \alpha_2)}, \tilde{b} \rangle d\text{vol}_{g_0}$$

for all anti-holomorphic 1-forms \tilde{b} vanishing to third order at

Let $b \in x^J L^2(M_0)$ for some $J \in \mathbb{R}$ be an antiholomorphic 1-form which vanishes at $\{p_1, \dots, p_n\}$. By Lemma 2.4, it can be written as the sum $(b - \tilde{b}) + \tilde{b}$ where $(b - \tilde{b}) \in x^{\tilde{J}} L^2(M_0)$ vanishes to third order at $\{p_0, \dots, p_n\}$ and $\tilde{b} \in x^{\tilde{J}} L^2(M_0)$ vanishes to third order at $\{\tilde{p}_0, \dots, \tilde{p}_m\}$. Linearity then implies that

$$0 = \int_{M_0} \langle \bar{\partial} e^{i(\alpha_1 - \alpha_2)}, b \rangle d\text{vol}_{g_0}$$

for all antiholomorphic 1-forms $b \in x^J L^2(M_0)$ vanishing to third order at $\{p_1, \dots, p_n\}$. Proceeding as such for the points p_1, \dots, p_n we can remove all the vanishing conditions placed upon b and the orthogonality condition (43) holds for all antiholomorphic 1-forms. The existence of Ψ satisfying the asymptotic condition (38) is thus proven by evoking Lemma 5.2 and observing that $U \in e^{-\gamma/x} H_b^m(M_0)$ for all $\gamma > 0$ iff $U \in e^{-\gamma/x} H^m(M_0)$ for all $\gamma > 0$.

To see that the holomorphic function Ψ is non-vanishing, we interchange the indices to deduce that there exists holomorphic functions $\Psi_{1,2}$ and $\Psi_{2,1}$ on M_0 which satisfies condition (38) for $e^{i(\alpha_1-\alpha_2)}$ and $e^{i(\alpha_2-\alpha_1)}$ respectively. Considering the product $\Psi_{1,2}\Psi_{2,1}$ and using condition (38) we see that the product is actually the constant function 1. \square

If α_j are the functions constructed in Proposition 2.2, it is convenient make the definition $F_{A_1} := e^{i\alpha_1}$ and $F_{A_2} := \Psi e^{i\alpha_2}$ where Ψ is the holomorphic function constructed in Proposition 5.1. Following the construction of Ψ and using the fact that $H^{3+\epsilon_0}(M_0) \subset W^{2,\infty}(M_0)$ one has the following useful expression for F_{A_2}

Lemma 5.4. We have that $F_{A_2} = e^{i\alpha_1}(1+t)$ with t bounded uniformly away from -1 and belongs to the space $e^{-\gamma/x}W^{2,\infty}(M_0)$ for all $\gamma > 0$.

6. CONSTRUCTION OF CGO SOLUTIONS

We construct special solutions to $(L_{X,V} - \lambda^2)u = 0$. To this end it is convenient to write the differential operator in terms $\bar{\partial}$ and its adjoint. Namely, if α is a function such that $\bar{\partial}\alpha = A$ then one can write

$$L_{X,V} = e^{-i\bar{\alpha}}\bar{\partial}^*|e^{-i\alpha}|^2\bar{\partial}e^{i\alpha} + Q$$

for $Q = |X|^2 + V + dX \in e^{-\gamma/x}L^\infty$ for all $\gamma \in \mathbb{R}$. We would like to study the existence of such functions α with suitable behaviour near the ends.

If M_0 is a surface with N Euclidean ends, consider its N point compactification M by adding the points $\{e_1, \dots, e_N\}$ at the ends. Around each e_j introduce holomorphic coordinate z and write $z = xe^{i\theta}$.

6.1. Constructing CGO of Type I. We are now in a position to construct a family CGO which will be useful for recovering interior information. Let b be a section of $T_{0,1}^*M$ belonging to $\text{Ker}(\bar{\partial}^*)$ with poles contained in the set $\{e_1, \dots, e_N\}$ and $\Phi = \phi + i\psi$ be a morse holomorphic function on M_0 with poles of the form $\frac{C_j}{z}$ near e_j , $C_j \neq 0$. Let F_A be a smooth function on M_0 which satisfies $F_A - e^{i\alpha} \in e^{-\gamma/x}W^{2,\infty}(M_0)$ for all $\gamma > 0$. If $\chi \in C_0^\infty(M_0)$ is a cutoff function which is 1 near all the critical points of Φ , consider the ansatz

$$u_0 = e^{\Phi/h}F_A^{-1}\bar{\partial}_J^{-1}(e^{-2i\psi/h}\chi|F_A|^2b) - h(1-\chi)e^{\bar{\Phi}/h}\bar{F}_A\frac{b}{2i\bar{\partial}\psi} \quad (45)$$

for $2 > J > 1$. Here we use the notation $(1-\chi)\frac{b}{\bar{\partial}\psi}$ to denote the function satisfying $\bar{\partial}\psi(1-\chi)\frac{b}{\bar{\partial}\psi} = (1-\chi)b$. It is well-defined since ψ has no critical points on the support of $(1-\chi)$.

By writing $L_{X,V} = \bar{F}_A\bar{\partial}^*|F_A|^{-2}\bar{\partial}F_A + Q$ direct computation yields that

$$(L_{X,V} - \lambda^2)u_0 = -\lambda^2u_0 + he^{\bar{\Phi}/h}e^{-\gamma/x}L^2(M_0)$$

for all $\gamma > 0$. Consequently we can use the estimates we established in Lemma 2.8 and the expression for u_0 to obtain

$$(L_{X,V} - \lambda^2)u_0 = e^{\Phi/h}F_A^{-1}O_{x^{-J}L^2}(h^{\frac{1}{2}+\epsilon}) + he^{\bar{\Phi}/h}\bar{F}_Ax^{-J'}L^2 \quad (46)$$

for some $J' > 0$ and $2 > J > 1$. By Proposition 3.5 we can solve for the remainder r so that $(L_{X,V} - \lambda^2)(u_0 + e^{\varphi/h}r) = 0$ with r satisfying the estimate $\|e^{-\gamma_0/x}r\| + h\|e^{-\gamma_0/x}dr\| \leq Ch^{1+\epsilon}$ for some $\gamma_0 > 0$. We summarize this discussion in the following Proposition.

Proposition 6.1. There exists solutions to $(L_{X,V} - \lambda^2)u = 0$ of the form $u = u_0 + e^{\phi/h}r$ where u_0 is given by (45) and r satisfies the estimate

$$\|e^{-\gamma_0/x}r\| + h\|e^{-\gamma_0/x}dr\| \leq Ch^{1+\epsilon}$$

for some $\gamma_0 > 0$.

6.2. Constructing CGO of Type II. Let $\Phi = \phi + i\psi$ be a holomorphic Morse function which has critical points $\{p_0, \dots, p_n\}$ and expansion $\Phi = \frac{C_j}{z}$ for $C_j \neq 0$ near the ends e_j for $j = 1, \dots, N$. Let a be a holomorphic function in $x^{-J}L^2(M_0)$ for some $J \in \mathbb{R}_+ \setminus \mathbb{Z}$ and which vanishes at $\{p_1, \dots, p_n\}$ but does not vanish at p_0 . We see then that

$$(L_{X,V} - \lambda^2)e^{\Phi/h}e^{-i\alpha}a = (Q - \lambda^2)e^{\Phi/h}e^{-i\alpha}a$$

and this motivates us to seek r_1 solving

$$(L_{X,V} - \lambda^2)e^{\Phi/h}r_1 = -(Q - \lambda^2)e^{\Phi/h}e^{-i\alpha}a + e^{\Phi/h}e^{-i\alpha}O_{x^{-J}L^2}(h|\log h|).$$

To this end, let G be the operator of Lemma 2.6, mapping continuously $x^{-J+1}L^2(M_0)$ to $x^{-J-1}L^2(M_0)$. Then clearly $\partial^*\partial G = Id$ when acting on $x^{-J+1}L^2$.

First, we will search for r_1 satisfying

$$e^{-2i\psi/h}|e^{i\alpha}|^2\partial e^{i\bar{\alpha}}e^{2i\psi/h}r_1 = -\partial G(a(Q - \lambda^2)) + \omega + O_{x^{-J}H^1}(h|\log h|) \quad (47)$$

with $\omega \in x^{-J}L^2(M_0)$ a holomorphic 1-form on M_0 and $\|r_1\|_{x^{-J}L^2} = O(h)$. Indeed, using the fact that Φ is holomorphic we have

$$e^{-\Phi/h}(L_{X,V} - \lambda^2)e^{\Phi/h} = e^{-i\alpha}\partial^*e^{-2i\psi/h}|e^{i\alpha}|^2\partial e^{i\bar{\alpha}}e^{2i\psi/h} + \bar{Q} - \lambda^2$$

for some smooth superexponentially decaying function Q and applying $e^{-i\alpha}\partial^*$ to (47), this gives

$$e^{-\Phi/h}(L_{X,V} - \lambda^2)e^{\Phi/h}r_1 = -ae^{-i\alpha}(Q - \lambda^2) + e^{-i\alpha}O_{x^{-J}L^2}(h|\log h|).$$

Writing $-\partial G(a(Q - \lambda^2)) =: c(z)dz$ in local complex coordinates, $c(z)$ is $C^{2,\gamma}$ by elliptic regularity and we have $2i\partial_{\bar{z}}c(z) = a(Q - \lambda^2)$, therefore $\partial_z\partial_{\bar{z}}c(p') = \partial_{\bar{z}}^2c(p') = 0$ at each critical point $p' \neq p_0$ by construction of the function a . Therefore, we deduce that at each critical point $p' \neq p_0$, $c(z)$ has Taylor series expansion $\sum_{j=0}^2 c_j z^j + O(|z|^{2+\gamma})$. That is, all the lower order terms of the Taylor expansion of $c(z)$

around p' are polynomials of z only. By Lemma 2.5, and possibly by taking J larger, there exists a holomorphic function $f \in x^{-J}L^2$ such that $\omega := \partial f$ has Taylor expansion equal to that of $\partial G(a(Q - \lambda^2))$ at all critical points $p' \neq p_0$ of Φ . We deduce that, if $\beta := -\partial G(a(Q - \lambda^2)) + \omega = \beta(z)dz$, we have

$$\begin{aligned} |\partial_{\bar{z}}^m \partial_z^\ell \beta(z)| &= O(|z|^{2+\gamma-\ell-m}), \quad \text{for } \ell + m \leq 2, \text{ at critical points } p' \neq p_0 \\ |\beta(z)| &= O(|z|), \quad \text{if } p' = p_0. \end{aligned} \quad (48)$$

Now, we let $\chi_1 \in C_0^\infty(M_0)$ be a cutoff function supported in a small neighbourhood U_{p_0} of the critical point p_0 and identically 1 near p_0 , and $\chi \in C_0^\infty(M_0)$ is defined similarly with $\chi = 1$ on the support of χ_1 . We will construct r_1 to be a sum $r_1 = r_{11} + hr_{12}$ where r_{11} is a compactly supported approximate solution of (47) near the critical point p_0 of Φ and r_{12} is correction term supported away from p_0 . We define locally in complex coordinates centered at p_0 and containing the support of χ

$$r_{11} := \chi e^{-2i\psi/h} e^{-i\bar{\alpha}} R(|e^{i\alpha}|^{-2} e^{2i\psi/h} \chi_1 \beta) \quad (49)$$

where $Rf(z) := -(2\pi i)^{-1} \int_{\mathbb{R}^2} \frac{1}{\bar{z}-\xi} f d\bar{\xi} \wedge d\xi$ for $f \in L^\infty$ compactly supported is the classical Cauchy operator inverting locally ∂_z (r_{11} is extended by 0 outside the neighbourhood of p_0). The function r_{11} is in $C^{3,\gamma}(M_0)$ and we have

$$\begin{aligned} e^{-2i\psi/h} |e^{i\alpha}|^2 \partial(e^{2i\psi/h} e^{i\bar{\alpha}} r_{11}) &= \chi_1 (-\partial G(a(Q - \lambda^2)) + \omega) + \eta \\ \text{with } \eta &:= e^{-2i\psi/h} e^{-i\bar{\alpha}} R(|e^{i\alpha}|^{-2} e^{2i\psi/h} \chi_1 \beta) \partial \chi. \end{aligned} \quad (50)$$

We then construct r_{12} by observing that b vanishes to order $2 + \gamma$ at critical points of Φ other than p_0 (from (48)), and $\partial \chi = 0$ in a neighbourhood of any critical point of ψ , so we can find r_{12} satisfying

$$2ie^{i\alpha} r_{12} \partial \psi = (1 - \chi_1) \beta. \quad (51)$$

This is possible since both $\partial \psi$ and the right hand side are valued in $T_{1,0}^* M_0$ and $\partial \psi$ has finitely many isolated zeroes on M_0 : r_{12} is then a function which is in $C^{2,\gamma}(M_0 \setminus P)$ where $P := \{p_1, \dots, p_n\}$ is the set of critical points other than p_0 , it extends to a function in $C^{1,\gamma}(M_0)$ and it satisfies in local complex coordinates z at each p_j , $j = 1, \dots, n$

$$|\partial_z^j \partial_{\bar{z}}^k r_{12}(z)| \leq C |z|^{1+\gamma-j-k}, \quad j+k \leq 2$$

by using also the fact that $\partial \psi$ can be locally be considered as a smooth function with a zero of order 1 at each p_j . Moreover $\beta \in x^{-J} H^2(M_0)$ thus $r_1 \in x^{-J} H^2(M_0)$ and we have

$$e^{-2i\psi/h} |e^{i\alpha}|^2 \partial(e^{i\bar{\alpha}} e^{2i\psi/h} r_1) = -\partial G(a(Q - \lambda^2)) + \omega + h |e^{i\alpha}|^2 \partial e^{i\bar{\alpha}} r_{12} + \eta.$$

Lemma 6.1. The following estimates hold true

$$\begin{aligned} \|\eta\|_{H^2(M_0)} &= O(|\log h|), \quad \|\eta\|_{H^1(M_0)} \leq O(h |\log h|), \quad \|x^J \partial r_{12}\|_{H^1(M_0)} = O(1), \\ \|x^J e^{i\alpha} r_1\|_{L^2} &= O(h), \quad \|x^J e^{i\alpha} (r_1 - h \tilde{r}_{12})\|_{L^2} = o(h) \end{aligned}$$

where \tilde{r}_{12} solves $2ie^{i\alpha} \tilde{r}_{12} \partial \psi = \beta$.

Proof. The proof is exactly the same as the proof of Lemma 4.2 in [10], except that one needs to add the weight x^J to have bounded integrals. \square

As a direct consequence, we have

Corollary 6.1. With $r_1 = r_{11} + hr_{12}$, there exists $J > 0$ such that

$$\|e^{i\alpha} e^{-\Phi/h} (L_{X,V} - \lambda^2) e^{\Phi/h} (a + r_1)\|_{x^{-J} L^2(M_0)} = O(h |\log h|).$$

Now we can apply Proposition 3.5 to obtain solutions to $(L_{X,V} - \lambda^2)u = 0$ of the form

$$u = e^{\Phi/h}(a + r_1) + e^{\varphi/h}r_2 \quad (52)$$

with r_2 satisfying the estimates

$$\|e^{-\gamma_0/x}r_2\| + h\|e^{-\gamma_0/x}dr_2\| \leq Ch^{1+\frac{1}{2}}|\log h|$$

for some $\gamma_0 > 0$.

7. CONJUGATION FACTORS AND AN INTEGRAL IDENTITY

We begin by defining the functions F_{A_1} and F_{A_2} as

$$F_{A_1} := e^{i\alpha_1} \quad \text{and} \quad F_{A_2} := \Psi e^{i\alpha_2}, \quad (53)$$

where α_j are the solutions to $\bar{\partial}\alpha_j = A_j$, given by Proposition 2.2 and where Ψ is the holomorphic function given by Proposition 5.1 so that F_{A_2} has the expression given by Lemma 5.4

We proceed by first deriving an appropriate system and from this an integral identity. Consider the equation

$$(L_{X_j,V_j} - \lambda^2)u_j = 0, \quad \text{on} \quad M_0, \quad (54)$$

$j = 1, 2$. This equation can be rewritten by means of the above definitions in the form

$$(2F_{A_j}^{-1}\bar{\partial}^*F_{A_j}F_{A_j}^{-1}\bar{\partial}F_{A_j} - \lambda^2 + Q_j)u_j = 0,$$

where $Q_j := \star dX_j + V_j$. Using this and by defining the function and 1-form

$$\tilde{u}_j := F_{A_j}u_j, \quad \tilde{\omega} := F_{A_j}^{-1}\bar{\partial}\tilde{u}_j = |F_{A_j}|^{-2}\bar{\partial}\tilde{u}_j,$$

and further setting

$$D := \begin{pmatrix} 0 & \bar{\partial}^* \\ \bar{\partial} & 0 \end{pmatrix}, \quad \mathcal{A}_j := \begin{pmatrix} (Q_j - \lambda^2)|\bar{F}_{A_j}|^2 & 0 \\ 0 & -|F_{A_j}|^2 \end{pmatrix}, \quad U_j := \begin{pmatrix} \tilde{u}_j \\ \tilde{\omega}_j \end{pmatrix},$$

one sees that equation (54) is then equivalent to the system

$$(D + \mathcal{A}_j)U_j = 0, \quad \text{on} \quad M_0.$$

In order to derive the integral identity we define an exhaustion of M_0 , given by the sets $M_R := \tilde{M} \cup \tilde{E}_{1,R} \cup \dots \cup \tilde{E}_{N,R}$, where $\tilde{M} \setminus \cup_j E_j$ and $\tilde{E}_{j,R} \simeq B(0, R) \setminus B(0, 1)$, $R > 1$. Consider the solutions U_j of $(D + \mathcal{A}_j)U_j = 0$, $j = 1, 2$. Letting $W := U_1 - U_2$, we have that

$$\int_{M_R} \langle (D + \mathcal{A}_1)W, U_1 \rangle = \int_{M_R} \langle (\mathcal{A}_2 - \mathcal{A}_1)U_2, U_1 \rangle,$$

where $\langle U_1, U_2 \rangle := \langle \tilde{u}_1, \tilde{u}_2 \rangle + \langle \tilde{\omega}_1, \tilde{\omega}_2 \rangle$ where inner products on the right side are the ones induced by the metric. By integrating by parts the left hand side and using the fact that $(D - \mathcal{A}_1)U_1 = 0$, we have that

$$\int_{M_R} \langle (\mathcal{A}_2 - \mathcal{A}_1)U_2, U_1 \rangle = \int_{\partial M_R} \iota^*((\tilde{u}_1 - \tilde{u}_2) \star \overline{\tilde{\omega}_1} - \star(\tilde{\omega}_1 - \tilde{\omega}_2)\overline{\tilde{u}_1}), \quad (55)$$

where $\iota: \partial M_R \rightarrow M_0$ is the inclusion map. The boundary ∂M_R can be decomposed into components that are contained in the ends E_j . The integration set ∂M_R can moreover be considered to consist of the set $\{z \in \mathbb{C}: |z| = R\}$ in each end.

We now let $u_j \in x^{-\tau} L^2$, $\tau > \frac{1}{2}$ be the scattering solutions given by means of the Poisson operator, i.e. $u_j = P_j(\lambda)g$, where $g \in C^\infty(\partial M_0)$ and consider how the boundary integral in (55) behaves, when $R \rightarrow \infty$. By Proposition 4.1 we have the following asymptotics for the scattering solutions,

$$u_j = P_j(\lambda)g = c_\lambda r^{-\frac{1}{2}}[e^{i\lambda r}(S_{X_j, V_j}(\lambda)g)(\theta)) + ie^{-i\lambda r}g(\theta)] + \mathcal{R},$$

where $z = re^{i\theta}$ and $|\mathcal{R}| + |\nabla \mathcal{R}| \leq Cr^{-\frac{3}{2}}$.

To see that the boundary integral in (55) vanishes in the limit, we firstly note that

$$(\tilde{u}_1 - \tilde{u}_2)\overline{\tilde{\omega}_1} = (F_{A_1}u_1 - F_{A_2}u_2)\overline{\tilde{\omega}_1} = (F_{A_1}(u_1 - u_2) + (F_{A_1} - F_{A_2})u_2)\overline{\tilde{\omega}_1}$$

The term containing $F_{A_1} - F_{A_2}$ decays super exponentially thanks to Proposition 5.1 and will therefore not contribute to the integral in (55) in the limit. We have moreover that

$$\begin{aligned} F_{A_1}(u_1 - u_2)\overline{\tilde{\omega}_1} &= F_{A_1}(u_1 - u_2)|F_{A_1}|^{-2}(\overline{F_{A_1}}\partial\overline{u_1} + \overline{u_1}\partial\overline{F_{A_1}}) \\ &= (u_1 - u_2)\partial\overline{u_1} + O(e^{-r}). \end{aligned}$$

The scattering matrices are equal for the potentials, i.e. $S_{X_1, V_1}(\lambda)g = S_{X_2, V_2}(\lambda)g$. This together with the above asymptotics for u_j imply that the above expression is $O(r^{-3/2})$. It follows the last term in the integral in (55) vanishes in the limit $R \rightarrow \infty$.

To handle the $\tilde{\omega}_1 - \tilde{\omega}_2$ term in (55), we argue similarly. First note that the derivative has the expansion

$$\bar{\partial}u_j = c_\lambda r^{-\frac{1}{2}}[i\lambda e^{i\lambda r}S_{X_j, V_j}(\lambda)g(\theta)) + \lambda e^{-i\lambda r}g(\theta)] + O(r^{-\frac{3}{2}}).$$

Secondly we have that

$$\tilde{\omega}_j = |F_{A_j}|^{-2}\bar{\partial}F_{A_j}u_j - |F_{A_j}|^{-2}F_{A_j}\bar{\partial}u_j = -|F_{A_j}|^{-2}F_{A_j}\bar{\partial}u_j + O(e^{-r}),$$

It follows that

$$\begin{aligned} (\tilde{\omega}_1 - \tilde{\omega}_2)\overline{\tilde{u}_1} &= \left(\frac{F_{A_2}}{|F_{A_2}|^2}\bar{\partial}u_2 - \frac{F_{A_1}}{|F_{A_1}|^2}\bar{\partial}u_1\right)\overline{F_{A_1}u_1} + O(e^{-r}) \\ &= ((\overline{F_{A_1}} - \overline{F_{A_2}})\bar{\partial}u_2 + \overline{F_{A_2}}(\bar{\partial}u_2 - \bar{\partial}u_1))\frac{F_{A_2}}{|F_{A_2}|^2}\overline{u_1} + O(e^{-r}), \end{aligned}$$

the first term decays super exponentially by Proposition 5.1. In the second term the F_{A_2} 's cancel, and we can thus use the asymptotics of $\bar{\partial}u_j$ together with the fact that $S_{X_1, V_1}(\lambda)g = S_{X_2, V_2}(\lambda)g$ to see that it is $O(r^{-3/2})$. This implies that the integral containing the $\tilde{\omega}_1 - \tilde{\omega}_2$ term in (55), will vanish, when $R \rightarrow \infty$.

By taking the limit $R \rightarrow \infty$ in (55) we obtain hence that

$$\int_{M_0} \langle (\mathcal{A}_2 - \mathcal{A}_1)U_2, U_1 \rangle = 0, \quad (56)$$

when U_1 and U_2 are made up of scattering solutions. As a consequence of the density result of Proposition 4.2 we can extend this to all exponentially growing solutions of (54), which is the content of the following Lemma.

Lemma 7.1. Assume that $S_{X_1, V_1}(\lambda)g = S_{X_2, V_2}(\lambda)g$. Let $v, w \in e^{\gamma/x}H^1(M_0)$ for some $\gamma > 0$, be solutions of $(L_{X_1, V_1} - \lambda^2)v = 0$ and $(L_{X_2, V_2} - \lambda^2)w = 0$, on M_0 , then

$$\int_{M_0} \langle (|F_{A_1}|^{-2} - |F_{A_2}|^{-2})\bar{\partial}v, \bar{\partial}w \rangle + \frac{1}{2} \langle (Q_1|F_{A_1}|^2 - Q_2|F_{A_2}|^2)\tilde{v}, \tilde{w} \rangle = 0,$$

Proof. Proposition 4.2 implies that we can pick two sequences of scattering solutions (v_k) and (w_k) , s.t. $v_k \rightarrow v$, and $w_k \rightarrow w$ in the $e^{\gamma'/x}L^2$ -norm. It follows that $\tilde{v}_k \rightarrow \tilde{v}$, and $\tilde{w}_k \rightarrow \tilde{w}$ in the $e^{\gamma'/x}L^2$ -norm and moreover that $\bar{\partial}\tilde{v}_k \rightarrow \bar{\partial}\tilde{v}$, and $\bar{\partial}\tilde{w}_k \rightarrow \bar{\partial}\tilde{w}$ in the $e^{\gamma'/x}H^{-1}$ -norm.

Use the abbreviations $\mu_F := |F_{A_1}|^{-2} - |F_{A_2}|^{-2} \in e^{-\gamma/x}W^{1,\infty}(M_0)$ and $\mu_Q := \frac{1}{2}(Q_1|F_{A_1}|^2 - Q_2|F_{A_2}|^2) \in e^{-\gamma/x}W^{1,\infty}(M_0)$. Now suppose first that u is a scattering solution to $(L_{X_2, V_2} - \lambda^2)u = 0$. Since the claim holds for scattering solutions, because of (56) we have that

$$\int_{M_0} \langle \mu_F \bar{\partial}v, \bar{\partial}u \rangle + \langle \mu_Q \tilde{v}, \tilde{u} \rangle = \int_{M_0} \langle \mu_F \bar{\partial}(\tilde{v} - \tilde{v}_k), \bar{\partial}u \rangle + \langle \mu_Q(\tilde{v} - \tilde{v}_k), \tilde{u} \rangle \quad (57)$$

We can estimate the term involving μ_Q by

$$\int_{M_0} \langle \mu_Q(\tilde{v} - \tilde{v}_k), \tilde{u} \rangle \leq \|e^{\gamma'/x}\mu_Q\|_{L^\infty} \|e^{-\gamma'/x}(\tilde{v} - \tilde{v}_k)\|_{L^2} \|\tilde{u}\|_{L^2} \rightarrow 0, \quad (58)$$

as $k \rightarrow \infty$. To handle the other term in (57) we write

$$\begin{aligned} \langle \mu_F \bar{\partial}(\tilde{v} - \tilde{v}_k), \bar{\partial}u \rangle &= (|F_{A_1}|^{-2} - |F_{A_2}|^{-2})F_{A_1}\bar{F}_{A_2} \\ &\quad (iA_1(v - v_k) - \bar{\partial}(v - v_k))(-i\bar{A}_2\bar{u} - \partial\bar{u}). \end{aligned}$$

Furthermore we have that

$$\mu := (|F_{A_1}|^{-2} - |F_{A_2}|^{-2})F_{A_1}\bar{F}_{A_2} = ((F_{A_2} - F_{A_1})\bar{F}_{A_2} + F_{A_1}(\bar{F}_{A_2} - \bar{F}_{A_1}))/F_{A_2}\bar{F}_{A_1}.$$

Proposition 5.1 implies that μ is super exponentially decaying. To estimate the first term on the right hand side of (57) we write

$$\begin{aligned} \int_{M_0} \langle \mu_F \bar{\partial}(\tilde{v} - \tilde{v}_k), \bar{\partial}u \rangle &= \int_{M_0} \mu(iA_1(v - v_k) - \bar{\partial}(v - v_k))(-i\bar{A}_2\bar{u} - \partial\bar{u}) \\ &\leq \|A_1(v - v_k)\|_{L^2} \|A_2\bar{u}\|_{L^2} \\ &\quad + \|x^{-\alpha}A_1(v - v_k)\|_{L^2} \|x^\alpha\partial\bar{u}\|_{L^2} \\ &\quad + \|\mu\bar{\partial}(v - v_k)\|_{H^{-1}} \|A_2\bar{u}\|_{H^1} \\ &\quad + \|x^{-\alpha}\mu\bar{\partial}(v - v_k)\|_{H^{-1}} \|x^\alpha\partial\bar{u}\|_{H^1} \\ &\rightarrow 0, \end{aligned}$$

as $k \rightarrow \infty$, where we used the fact that $u \in x^{-\alpha}H^2(M_0)$, $\alpha > 0$ by elliptic regularity. It follows

$$\int_{M_0} \langle \mu_F \bar{\partial}v, \bar{\partial}u \rangle + \langle \mu_Q \tilde{v}, \tilde{u} \rangle = 0. \quad (59)$$

We can now repeat this form of argument to obtain the above equation with the scattering solution u replaced by w , thus proving the claim. \square

8. GAUGE EQUIVALENCE

The goal of this section is to prove the gauge equivalence statement of Theorem 1.1:

Proposition 8.1. If $S_{X_1, V_1}(\lambda) = S_{X_2, V_2}(\lambda)$ for a fixed $\lambda \in \mathbb{R} \setminus \{0\}$ then there exists a unitary function Θ such that $X_1 - X_2 = d\Theta/\Theta$.

Let $\Phi = \varphi + i\psi$ be a Morse holomorphic function given by Lemma 2.1 and Corollary 2.1, and let $\{p_0, \dots, p_n\}$ be its critical points. Let b be an antiholomorphic 1-form chosen so that

$$b(p_1) = \dots = b(p_n) = 0, \quad b(p_0) \neq 0. \quad (60)$$

Such antiholomorphic 1-forms are given by Lemma 2.4. Proposition 6.1 gives u_j , $j = 1, 2$ solving $(L_{X_j, V_j} - \lambda^2)u_j = 0$, and are of the form

$$u_1 := u_{0,1} + e^{\varphi/h} r_1, \quad u_2 := u_{0,2} + e^{-\varphi/h} r_2$$

where $u_{0,j}$ is given by (45) where $u_{0,1}$ is constructed with phase Φ while $u_{0,2}$ is constructed with phase $-\Phi$.

Let $\tilde{u}_j := F_{A_j} u_j$ and direct computation shows that

$$\bar{\partial} \tilde{u}_j = \bar{\partial}(F_{A_j} u_j) = e^{(-1)^{j-1} \bar{\Phi}/h} |F_{A_j}|^2 b + h e^{(-1)^{j-1} \bar{\Phi}/h} |F_{A_j}|^2 R_{0,j} + \bar{\partial}(F_{A_j} e^{(-1)^{j-1} \varphi/h} r_j) \quad (61)$$

where $R_{0,j} \in e^{\gamma_0/x} W^{1,\infty}(M_0)$ for some $\gamma_0 > 0$.

Plugging in the expression (61) into the identity given by Lemma 7.1 we obtain

$$\begin{aligned} o(h) &= \int_{M_0} (|F_{A_1}|^2 - |F_{A_2}|^2) \langle e^{\bar{\Phi}/h} (b + h R_{0,1}), e^{-\bar{\Phi}/h} (b + h R_{0,2}) \rangle \\ &+ \int_{M_0} (|F_{A_1}|^2 - |F_{A_2}|^2) \langle e^{\bar{\Phi}/h} (b + h R_{0,1}), |F_{A_2}|^{-2} \bar{\partial}(F_{A_2} e^{-\varphi/h} r_2) \rangle \\ &+ \int_{M_0} (|F_{A_1}|^2 - |F_{A_2}|^2) \langle |F_{A_1}|^{-2} \bar{\partial}(F_{A_1} e^{\varphi/h} r_1), e^{-\bar{\Phi}/h} (b + h R_{0,2}) \rangle \\ &+ \int_{M_0} (|F_{A_1}|^2 - |F_{A_2}|^2) \langle |F_{A_1}|^{-2} \bar{\partial}(F_{A_1} e^{\varphi/h} r_1), |F_{A_2}|^{-2} \bar{\partial}(F_{A_2} e^{-\varphi/h} r_2) \rangle. \end{aligned} \quad (62)$$

Note that Lemma 5.4 ensures $(|F_{A_1}|^2 - |F_{A_2}|^2) \in e^{-\gamma/x} W^{1,\infty}(M_0)$ for all $\gamma > 0$.

We need to show that everything on the right-side of (62) is $o(h)$ except the principal term $\int_{M_0} (|F_{A_1}|^2 - |F_{A_2}|^2) e^{-2i\psi/h} |b|^2$. This comes by direct computation using stationary phase for terms not involving $\bar{\partial}(F_{A_j} e^{\pm \varphi/h} r_j)$. For terms which has the same form as

$$\int_{M_0} (|F_{A_1}|^2 - |F_{A_2}|^2) \langle e^{\bar{\Phi}/h} b, |F_{A_2}|^{-2} \bar{\partial}(F_{A_2} e^{-\varphi/h} r_2) \rangle$$

we can take advantage of the super-exponential decay of $(|F_{A_1}|^2 - |F_{A_2}|^2)$ and integrate-by-parts

$$\begin{aligned} &\int_{M_0} (|F_{A_1}|^2 - |F_{A_2}|^2) \langle e^{\bar{\Phi}/h} b, |F_{A_2}|^{-2} \bar{\partial}(F_{A_2} e^{-\varphi/h} r_2) \rangle \\ &= \int_{M_0} \langle \bar{\partial}^* (|F_{A_1}|^2 - |F_{A_2}|^2) |F_{A_2}|^{-2} e^{\bar{\Phi}/h} b, (F_{A_2} e^{-\varphi/h} r_2) \rangle. \end{aligned}$$

This can now be estimated using the bound for the remainder r_2 stated in Proposition 6.1 and is of order $o(h)$.

For the term

$$\int_{M_0} (|F_{A_1}|^2 - |F_{A_2}|^2) \langle |F_{A_1}|^{-2} \bar{\partial}(F_{A_1} e^{\phi/h} r_1), |F_{A_2}|^{-2} \bar{\partial}(F_{A_2} e^{-\phi/h} r_2) \rangle$$

we can again integrate-by-parts to move all the derivatives to terms involving r_2 :

$$\begin{aligned} & \int_{M_0} (|F_{A_1}|^2 - |F_{A_2}|^2) \langle \bar{\partial}(F_{A_1} e^{\phi/h} r_1), |F_{A_1} F_{A_2}|^{-2} \bar{\partial}(F_{A_2} e^{-\phi/h} r_2) \rangle \quad (63) \\ &= \int_{M_0} \langle F_{A_1} e^{\phi/h} r_1, \bar{\partial}^* (|F_{A_1}|^2 - |F_{A_2}|^2) |F_{A_1} F_{A_2}|^{-2} \bar{\partial}(F_{A_2} e^{-\phi/h} r_2) \rangle. \end{aligned}$$

Observe that by Proposition 6.1 and (46) we have

$$\begin{aligned} \|e^{-\phi/h} \Delta e^{\phi/h} r_2\|_{e^{\gamma_0/x} L^2} &\leq C(\|dr_2\|_{e^{\gamma_0/x} L^2} + h^{-1} \|r_2\|_{e^{\gamma_0/x} L^2} \\ &\quad + \|e^{-\phi/h} (L_{X_2, V_2} - \lambda^2) u_{0,2}\|_{e^{\gamma_0/x} L^2}) \\ &\leq o(1) \end{aligned}$$

and we see therefore that (63) is $o(h)$. We can conclude then that (62) indeed becomes

$$\int_{M_0} (|F_{A_1}|^2 - |F_{A_2}|^2) e^{2i\psi/h} |b|^2 = o(h). \quad (64)$$

We are now in a position to prove

Lemma 8.1. If $S_{X_1, V_1}(\lambda) = S_{X_2, V_2}(\lambda)$ for a fixed $\lambda \in \mathbb{R} \setminus \{0\}$ then for F_{A_1} and F_{A_2} chosen as in (53) one has $|F_{A_1}| = |F_{A_2}|$.

Proof. Let $p_0 \in M_0$ the critical point of a Morse meromorphic function $\Phi = \varphi + i\psi$ on $M_0 \cup \{e_1, \dots, e_N\}$ whose (non-removable) poles are all simple and form precisely the set $\{e_1, \dots, e_N\}$. From Proposition 2.1 we know that such points form a dense subset of M_0 . If $\{p_0, \dots, p_n\}$ are the critical points of Φ , choose antiholomorphic 1-form b satisfying condition (60) and apply stationary phase expansion to (64) we see that $|F_{A_1}(p_0)| = |F_{A_2}(p_0)|$. Since p_0 can be chosen over an dense subset of M_0 , the continuity of F_{A_j} completes the proof. \square

This Lemma leads immediate to the

Proof of Proposition 8.1 By Lemma 8.1 we can define the unitary function $\Theta := \frac{F_{A_1}}{F_{A_2}} = \frac{F_{A_2}}{F_{A_1}}$. Note that due to Lemma 5.4, $\Theta \in 1 + e^{-\gamma/x} W^{1,\infty}(M_0)$ for all $\gamma > 0$. We see that $\bar{\partial}\Theta = i\pi_{0,1}(X_1 - X_2)/\Theta$ while $\partial\Theta = i\pi_{1,0}(X_1 - X_2)/\Theta$. Adding the two identities together we obtain Proposition 8.1. \square

9. DETERMINING THE ZEROth ORDER TERM

In this section we complete the proof of Theorem 1.1 by proving

Proposition 9.1. If $S_{X_1, V_1}(\lambda) = S_{X_2, V_2}(\lambda)$ for a fixed $\lambda \in \mathbb{R} \setminus \{0\}$ then $V_1 = V_2$.

This will be accomplished with CGO of type II given by (65). To this end, let u_1 and u_2 be solutions of the from (65) with phase Φ and $-\Phi$ respectively:

$$u_1 = e^{\Phi/h}(a + r_1) + e^{\varphi/h} r_2, \quad u_2 = e^{-\Phi/h}(a + s_1) + e^{-\varphi/h} s_2$$

with r_2 and s_2 satisfying the estimates

$$\|e^{-\gamma_0/x} r_2\| + h \|e^{-\gamma_0/x} dr_2\| + \|e^{-\gamma_0/x} s_2\| + h \|e^{-\gamma_0/x} ds_2\| \leq Ch^{1+\frac{1}{2}} |\log h|$$

for some $\gamma_0 > 0$.

Note that since we have already shown in Proposition 8.1 that X_1 and X_2 are gauge equivalent, we may assume without loss of generality that they are actually

identical. Therefore, for the CGO u_1 and u_2 , the identity in Lemma 7.1 holds with $F_{A_1} = F_{A_2} = F_A = e^{i\alpha}$ to become

$$0 = \int_{M_0} |F_A|^2 \langle (Q_1 - Q_2) \tilde{u}_1, \tilde{u}_2 \rangle = \int_{M_0} |F_A|^4 \langle (Q_1 - Q_2) u_1, u_2 \rangle$$

where $\tilde{u}_j = F_A u_j$ and $Q_j = *dX_j + V_j$. We now plug in the expression for u_1 and u_2 into this identity. Using Lemma 6.1 and elementary estimates we obtain

$$o(h) = \int_{M_0} e^{2i\psi/h} (Q_1 - Q_2) |a|^2$$

Repeating the same argument as in proof of Lemma 8.1 we have that $Q_1 = Q_2$ on M_0 which implies that $V_1 = V_2$ and Proposition 9.1 is verified.

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